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Beyond written computation

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Beyond Written Computation

Edited by

Alistair McIntosh & Len Sparrow

MASTEC

**Edith Cowan University
Perth, Western Australia**

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All papers in this book were discussed at the Rottnest conference and subsequent changes were made by the authors based on comments and recommendations from the peer group who attended the conference. Attendees at the conference were:

Dr Mike Askew	King's College, London
Associate Professor Susie Groves	Deakin University
Associate Professor Alistair McIntosh	University of Tasmania
Dr Joanne Mulligan	Macquarie University
Dr Len Sparrow	Curtin University of Technology
Professor Kaye Stacey	University of Melbourne
Dr Paul Swan	Edith Cowan University
Professor Paul Trafton	University of Northern Iowa

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Foreword

"What's one and one and one and one and one and one and one and one and one and one?"

"I don't know," said Alice, "I lost count."

"She can't do addition," said the Red Queen.

(Lewis Carroll, Through the Looking Glass)

The ability to distinguish quantity is not a human prerogative. Some think that birds and even wasps can count (Dantzig, 1954). The idea of quantity exists without humans, but it appears doubtful that humankind can exist without the idea of quantity. Certainly since the coming of the human race, civilisations have developed and extended ways of representing number, of counting and of calculating. As they have developed these survival tools, they have developed ways of passing them on to their children and successors.

We are still engaged in that necessary task today. It never was, and it never can be, a static task with a final goal. The needs of society change: the methods of recording and of calculating are refined or take a quantum leap; and we change and evolve new modes and methods of imparting these skills to our children. There never was agreement. There never will be. The matter is far too important to allow for that.

This book seeks to take stock of where we are in this process, and where we should go. It looks briefly at the development of calculation and the progress of attempts to pass these skills to ever increasing proportions of our children. It charts the rise of the importance placed on written computation as the major skill required. It indicates and describes the pressures acting on this position and the questions raised as to its continuing relevance; and it shows examples of classrooms where attempts are being made to forge and deliver appropriate computational goals for the future. Finally, it discusses what needs to be done in the light of all this.

The history of the increasingly sophisticated ways that people have devised in order to record numbers of increasing size is outside the scope of this book. This book is concerned with calculations. It is important to distinguish between the two. Throughout most of history many people have found ways of recording numbers, including the results of calculations, by means of symbols. But throughout most of history people have not used those symbols for calculating. They have devised and refined a number of calculating devices, such as the abacus in its various forms.

Our most profound hope is that this book will not appear to be trying to impose a new orthodoxy on the reader, but that it will succeed in providing information and the opening up of the subject in such a way that the readers are able to make reasoned, informed and responsible judgments for themselves.

It is more important that people think, than that they think what we think.

It is easy and dangerous to generate heat. It is certainly more difficult and less sensational to generate light.

Let there be light!

Reference

Dantzig, T. (1954). *Number, the language of science*. New York: Macmillan.

Introduction

Where we are Today

Alistair McIntosh

University of Tasmania

The average child in an American elementary school receives some 650 hours of instruction in arithmetic. I arrive at this figure by assuming that children have a three-quarter hour mathematics lesson every day for 180 days per year for six years, and that arithmetic instruction is the topic for four out of the five lessons per week. There are several assumptions in there that you could question. But I think that it is a fair estimate and one which is not dissimilar to the practice in many countries.

The period of 650 hours is a large investment of time. Looked at one way, it is almost 40 000 minutes—and a minute is a long time for some children in a mathematics lesson. Looked at from another point of view, it is the equivalent of about twelve college courses—you could have two courses on number concepts, one on basic facts, one on each of the four operations, two on decimals, one on fractions, one on problem solving, and still have a whole course left over for revision. From any point of view, it is a substantial commitment of time, and, since time costs money, it is a large financial commitment on the part of the community.

That being so, two things are important: first, that an effective job is done; and second, that the job done is the right one. The community at large often queries the first point, mainly on anecdotal evidence, but seldom questions the second. In this book, we query both those assumptions: we ask whether what is being done is effective, but more important; we ask whether the goals of arithmetic in most primary/elementary schools are, any longer, the right goals. We suggest that what is being done is largely ineffective, not at all because of any lack of effort on the part of teachers, but because the goals themselves are unrealistic. More importantly, we argue that the goals themselves are not appropriate for the present generation of children; and we try to indicate what, in the opinion of the authors, are appropriate goals for elementary school arithmetic in today's society.

First we take an overview of where we are, and how we have arrived here; and then we chart in greater detail the conditions that are telling us to change direction. We will take some glimpses into classrooms and hear from teachers who have begun to make that change. And finally we will assess the implications of what we have described and discuss some options and practical possibilities for steps that we can take.

In ancient Greece, the reporter of bad tidings was usually killed (a practice incidentally which if applied to today's media reporters might brighten up our newspapers and TV screens considerably). We hope therefore that you will see the news we bring as, in the end, not bad, but positive and optimistic.

Before you reach the news in the later chapters of this book I want to address you, the dedicated elementary or primary schoolteacher. In this volume the term 'primary' refers to pre-secondary school—usually up to grade or Year 6 (or 7 in some school systems). In mathematics, you teach number for three or four out of five days each week. Your main aim in number (I still prefer to call it arithmetic) is, at an appropriate age, to teach children the four rules of number, pencil and paper arithmetic, with understanding, through the use of materials.

Pencil and paper arithmetic, at an appropriate age, with understanding, through the use of materials

I want to take these phrases and consider some of the issues and advice underpinning each of them in turn. Are they still relevant? Are they still appropriate for today's children? Are they of use to classroom teachers?

With understanding?

Well, let's face it, that's not what all teachers try to do. Not all, at an appropriate age, with understanding, and through the use of materials. I remember going in to a Year 6 classroom to supervise a student teacher. As I entered, on my right the teacher, a dynamic bundle, was 'foghorn' a test to the main body of children, while at the back my student was working at a table with five students. My student was explaining to me that these were children who were having great difficulties with subtraction, and she was helping them use Base Ten materials to unravel the problems. Suddenly, there was relative silence, and I heard a determined clunk of footsteps approaching. The 'foghorn' stood before me and in a deafening whisper said: "She's not doing that understanding stuff is she? It only confuses them!"

Well, she is severely out of date. That argument was fought out in the 1920s and 1930s. While some argued, like the foghorn, that teaching for understanding was a waste of time, and that one should simply work through the list of things that children had to know, or know how to do, one after the other, as so many jobs to be done and ticked off, the views of those who argued that teaching without understanding was inefficient, since understanding aided and accelerated learning, prevailed. As William Brownell put it in 1935:

Within the teaching of arithmetic there is absolutely no place for the view of arithmetic as a heterogeneous mass of unrelated elements to be taught through repetition. The 'meaning' theory conceives of arithmetic as a closely-knit system of understandable ideas, principles and processes... The test of learning is not mere mechanical facility in 'figuring'. The true test is an intelligent grasp upon number relations and the ability to deal with arithmetic operations with proper comprehension of their mathematical as well as their practical significance. (p. 3)

These views were backed up by research findings such as those of Lyon and Reed, reported by Brownell and Moser in 1949. Lyon reported that his subjects required, on average, 93 minutes to learn 200 nonsense syllables, but only 10 minutes to learn 200 words of poetry. Reed found that it took his subjects, college students, an average of less than two minutes to learn 9 lines of simple prose narrative, but almost four and one half minutes to learn the same amount of difficult prose. Moreover, in retention his subjects recalled 49 out of 67 ideas in the simple narrative but only 11.5 out of 67 ideas in the complex narrative. Brownell and Moser themselves conducted the crucial piece of research in 1949 which established the superiority of decomposition taught with understanding as the most efficient subtraction method to teach in schools.

Thus, sense making from understanding in arithmetic is important. More meaningful learning can be achieved overall and in a shorter time if an emphasis is placed on children understanding the ideas and processes. But this is not new information for teachers and others interested in arithmetic.

At an appropriate age?

The Committee of Seven on Grade-Placement in Arithmetic began in 1926 and was still continuing their work in 1939. Their findings were the result of testing in 255 cities in 16 states, involving 1190 teachers and 30 744 children. They defined an appropriate age for teaching a subject as that when tests show that three quarters of the children can recall seventy percent of the material six weeks after teaching (Bidwell & Clason, 1970).

They used the concept of mental age as determined by intelligence tests, and among their findings were that children with a mental age of six to seven should receive “informal experiences to give children real concepts of number and space relations, without any systematic drills” (p. 577). Children with a mental age of eight to nine “can mechanically subtract three digits but we doubt whether such numbers have any real meaning for them”, while only at a mental age of thirteen to fourteen were children ready for fraction work involving “some addition and subtraction with unlike denominators” (p. 577).

Or take the research of John Biggs involving some 5000 children in 87 schools in England in 1967. Biggs (1967) was investigating the effects of teaching arithmetic by various means including traditional ‘chalk and talk’ and the use of various materials from Cuisenaire Rods to Multibase Arithmetic Blocks (MAB). From the data he collected he found that “there was no evidence that the amount of formal work done in the infants department [grades 1/2] bore any relationship to attainment later on in the junior school”. He concluded that “if thorough drilling in the infant school produces results that are little different from those that would be expected on the basis of the initial intelligence of the groups ... then it would seem that there is something to be said for doing work that is ... more interesting and that may be more rewarding in the long run” (p. 241).

In other words, formal number work in the first two years of school is, at best, a waste of time, and, more generally, we have tended to teach arithmetic skills at an inappropriately early age. It can of course be more than just a waste of time if it also produces negative attitudes on the part of children. But again, this is not new information for most teachers.

Through the use of materials?

From a great variety of studies I am most convinced by Biggs' painstaking, thoughtful and cautious study mentioned above (Biggs, 1967):

On the basis of the present study, then, we may conclude that, relative to a highly formally taught but otherwise strictly comparable group of children, the children taught in a manner designed to provide the maximum of appropriately structured experience with the minimum of the formal characteristics of rote learning will, when they have become used to the method, which seems to be in at least two or three years experience with it:

- possess a sounder grasp of the logical structure of arithmetic, which enables them to obtain higher scores not only on the concept test but also on the mechanical arithmetic test, and

- feel less worried and anxious about arithmetic and about schoolwork generally. (p. 242)

An equally strong argument for using materials for calculation can be drawn from the actual practicalities of the business of calculation through the ages. Through most of history, most people have used materials for calculating: we can think primarily of the abacus in all its forms: the ancient Roman abacus using grooves and pebbles; the medieval abacus using tokens as ‘counters’ and lines on a wooden surface; the various forms of bead abacus from the primitive ‘Russian’ to the Chinese abacus, and the more modern Japanese version, the Soroban. Research by Stigler and Perry (1990) has shown what I have long suspected that the use of the abacus gives its users particular facility with mental calculation through mental manipulation of the physical actions.

Apart from the abacus, there have been Napier’s Rods for multiplication and, within my own lifetime, the use of the slide rule, the mechanical hand calculator and, now, the electronic calculator and computer. Each is an object to aid calculation, as is the rather more awkward base ten materials or Multibase Arithmetic Blocks (MAB) as they are often known.

What we should bear in mind is this: the Hindu-Arabic numerals, as all other symbol systems, were devised to record the results of calculations. They need not also be used for the calculations themselves.

Pencil and paper arithmetic?

So far, so good. It is probably true to say that this is a fair and concise summary of the goal of the majority of thoughtful elementary teachers in developed countries today. But there is one phrase that I have not examined so far, and that is the first one.

Why the emphasis on pencil and paper arithmetic?

This is one tradition that seems like a universal truth, but isn’t. In fact, in America in the second half of the nineteenth century mental arithmetic maintained a status in schools that was equal to, if not greater than, written arithmetic, because of its supposed value in training the mind in memory and reasoning.

In the first half of the 20th century the acquisition of speed and accuracy in written computation may well have been a defensible major goal of universal education. An economic reason was that all bookkeeping, and there was a lot of it from small shopkeepers to banks, had to be done manually. Many films and stories provide an image of large, dusty, leather-bound ledgers and Dickensian clerks at desks with pens.

And we still have a lingering gut feeling that pencil and paper results are somehow more solid than answers obtained by mental calculation. Lewis Carroll—a pseudonym for Charles Dodgson—who was a mathematician (as well as the inventor of the folding map and one of the foremost amateur photographers of the Victorian age), expressed this neatly:

"How many days are there in a year?"

"Three hundred and sixty-five," said Alice.

"And how many birthdays have you?"

"One."

"And if you take one from three hundred and sixty-five, what remains?"

"Three hundred and sixty-four, of course."

Humpty Dumpty looked doubtful.

"I'd rather see that done on paper," he said.

But the reasons for paying less attention to written computation now are strong and need to be more widely known and discussed. Equally strong are the reasons for paying more, and more useful, attention, to mental computation. As the Cockcroft Report in England said in 1982:

The decline of mental and oral work within mathematics classrooms represents a failure to recognise the central place which working 'done in the head' occupies throughout mathematics. (p. 75)

One reason for placing less emphasis on written computation is that children, even at the height of their subjection to it, try to avoid using it:

One of the most remarkable things about these methods [standard written algorithms] is that they are used so little. In some research directed to quite other ends, D. A. Jones investigated the methods used by each of 80 11-year-olds to calculate $67 + 38$, $83 - 26$, 17×6 and $116 \div 4$. The questions were written in this form and the children were free to use written or mental methods. Over half of the 320 calculations were successfully completed by non-standard methods...Thus despite the heavy teaching of standard algorithms, they are not necessarily chosen for calculations of this difficulty. (Plunkett, 1979, p. 3)

Moreover, children appear to use their own invented mental methods more successfully than the written methods they have been taught. Nunes, Carraher and Schliemann (1987) interviewed 16 third-graders in Brazil, ranging in ages from 8 to 13 years. Each child was asked to solve ten problems in each of three conditions: (a) in a simulated store condition, in which the child played the role of the storekeeper and the experimenter the role of the customer, (b) embedded in word problems and (c) as computation exercises. In all cases the children were left free to use whatever calculation method they preferred. Figure 1 shows the percentage of correct responses given by procedure used (oral or mental) for each of the three conditions. In every case, the children used their own oral methods much more successfully than the written methods that they had been taught in school.

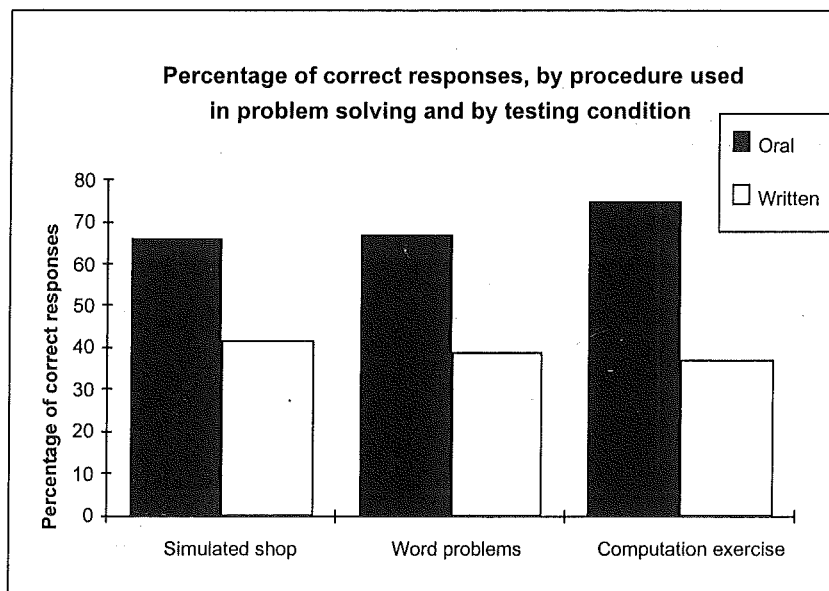


Figure 1: Performance on oral and written computation in different situations

Adults also seldom use written computation. Here is evidence from a study done in 1957, when electronic calculators were unknown.

The primary purpose of this study was to determine the relative importance of 'mental' and 'paper and pencil' mathematics...in the solutions of problems encountered in everyday non-occupational usage by adults.

Since 75 per cent of the uses reported were 'mental' and 25 per cent 'paper-and-pencil', in this study 'mental' uses outnumbered 'paper-and-pencil' uses in the ratio of 3 to 1. (Wandt & Brown 1957, p. 153)

Now that electronic calculating devices are universally cheap and available, it is reasonable to assume that the use of written calculation is proportionately even less common. In fact, McIntosh, Northcote and Sparrow (1999) some forty years later revisited the Wandt and Brown study and found, even in an age of cheap and readily available calculating devices, similar results. That is, there was a heavy emphasis on the use of personalised mental methods, some use of calculating devices and little use of formal, standard paper and pencil procedures. So the non-school world is full of mental computation, while the school world is filled with written computation. Surely this is not appropriate. Maier put the case thus:

If school is to be preparatory for life outside school, the school world ought to be as much like the non-school world as possible. In particular, young people in classrooms ought to do mathematics as it is done by folk in other parts of the world. School mathematics ought to emulate folk mathematics.

Woody Guthrie defined folk music as "music that folks sing". In the same way, folk mathematics is mathematics that folks do.... Folk mathematics is the way people handle the mathematics-related problems arising in everyday life.

... Some of the differences between school maths and folk maths are clear. One is that school maths is largely paper-and-pencil maths, while folk maths is not.

In folk maths, paper and pencil are a last resort. Yet they are the mainstay of school maths. (p. 22)

But the case against written computation lies not just in the fact that it is relatively useless, and a waste of valuable time. It is also harmful. First, it gives children a false idea of what one should do when calculating. Figure 2 presents some, and by no means rare, examples of what we persuade children to do when we concentrate on formal written algorithms. These instances were collected by Hope in 1986.

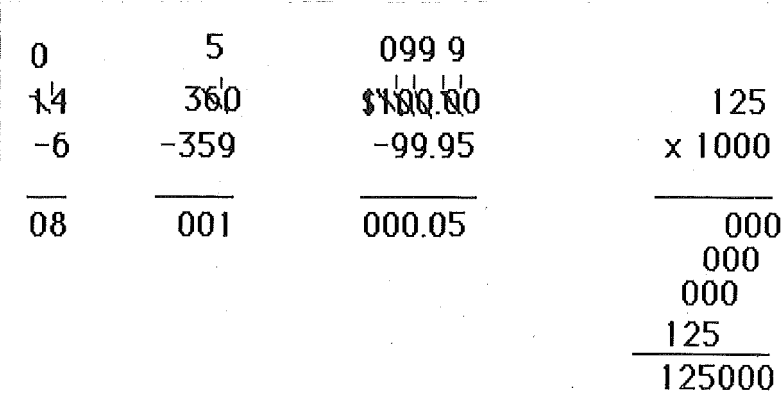


Figure 2: Examples of children's meaningless algorithms

In the first example, to calculate 14 subtract 6, the child has learned to write the calculation in vertical form, to say something like, '4 take 6 I cannot, take one ten and make it into ten ones, now 14 take 6 ...'. And of course that was the problem at the outset. The intervening activity has changed nothing and helped nothing. The interest lies in *how* to calculate $14 - 6$ *mentally*. The second example is dangerous in suggesting that counting and calculating are unrelated. The third is a case in which mental computation is obviously an easier solution strategy to use than the written algorithm; while the last example, with all its zeros, obscures the practical advantage of a base ten number system. Significantly, Hope (1986) followed these examples by observing:

When asked why she went through a lengthy approach to do a calculation that could have been done more easily 'in the head' one girl replied defensively, "That's what we have to do in school!" (p. 50)

Trafton (1986) summed up the situation in a clear and levelheaded way:

Estimation, mental computation, and calculators need to be accepted as legitimate computational methods. Students often feel that the estimation and mental-computation strategies they develop on their own must be kept from teachers because their use would not be considered 'proper'. (p. 2)

The effect of a distorting over-emphasis on formal computation is exemplified by the true story of a Japanese girl who was asked to make an estimate. "It is a wicked method," she said. Having been trained to give microscopically correct answers, being asked to give an inaccurate, and therefore 'wrong' answer was, to her, clearly immoral. We must be as careful about our effect on children's attitudes as we are about our concern for their technical skills. One defence made of teaching formal written algorithms is that this helps children to understand how numbers work. But a moment's thought makes this a very dubious proposition indeed.

A disadvantage of written methods is that the worker is not encouraged to think about the method he is using.... The practising of traditional written methods does not develop an awareness of the structure and properties of number. Contrary to this it will allow those with little understanding of place value to obtain right answers.... Now that we have calculators to give us right answers, the importance placed on traditional written methods must be in question. (Jones, 1988, p. 43)

Here is Hope voicing an opinion about which I have received a great deal of anecdotal support.

Some able children may lose part of their ability to calculate mentally as they begin to learn the algorithms taught in school.... An early emphasis on written algorithms may discourage the development of the ability to calculate mentally. (Hope 1987, p. 333)

I have had many acquaintances tell me that their child was keen on numbers before he or she started school and was quite adept at simple mental problems: but after school set in, the talk was of 'the way my teacher says you should do it' and the interest waned.

Edith Biggs, that wonderful lady, who stomped round the villages of England galvanising teachers into letting children do things and not just write things in mathematics lessons, said this:

This then is a crucial test of readiness for practice in written computation with tens and units: the ability to add two 2-digit numbers mentally by an efficient method. (Biggs, cited in Ewbank, 1977, p. 29)

And Ewbank, who gave this quote, continued:

... At first this looks like putting the cart before the horse. But on reflection I feel it makes good sense. (Ewbank, 1977, p. 29)

And she, and he, are right. When children have devised a way of adding two-digit numbers mentally, they show that they have understood both the problem and a method of solution. They understand sufficiently how numbers work. Then they can understand another way produced from outside, because they will be able to relate the new way to the way they have themselves constructed. Of course you can't decree which way they will use in future, theirs or yours.

Two further reasons for encouraging an emphasis on mental rather than written computation are that, first, we need mental computation constantly in order to help us make mental estimates; and, second, we need it in order to decide whether to trust or query results obtained on electronic calculators and electronic cash registers.

Finally, mental computation, if approached correctly, is more in tune with current practices in mathematics education. Let me explain this. I invite you to do two things. First, pause a moment and do this addition as a written calculation:

$$57 + 86$$

I imagine that almost all of you wrote 86 under 57, drew a line beneath, perhaps put an addition sign on the left of 86, and said '7 and 6 are 13, put down the 3 and carry the 1, 5 and 8 are 13 and 1 is 14. Answer 143'.

Now do this calculation mentally, without writing anything:

$$25 + 89$$

Some of you probably imagined doing mentally something very similar to what you did with the previous calculation. You may even have closed your eyes, and moved your hand in the air as though writing. But I suspect that most of you didn't. Perhaps you used one of the following methods:

$$20 + 80 = 100, 5 + 9 = 14, 100 + 14 = 114.$$

$$25 + 75 = 100, 100 + 14 = 114$$

$$89 + 11 = 100, 100 + 14 = 114$$

$$89 + 25 = 90 + 24 = 114$$

$$100 + 25 = 125, 125 - 11 = 114$$

These are among the many different ways people have told me they have performed this calculation mentally. What do they have in common? Well, first, they are all different. Second, they all work. Third, probably none of them were taught to you at school. Fourth, they were devised by you after considering closely, if only for an instant, the particular numbers involved, what you knew about them, how they related to other numbers (20 and 80 make 100, 89 is near 90 or 100 and so on) and how you could use what you know to make this particular calculation as simple as possible. You were being creative, active, concentrating on number relationships and problem-solving. Moreover, other people's methods for completing this calculation were of interest to you because they were different from your method.

How different this is from the atmosphere engendered by the written computation with its concern and anxiety. Did I get it right? Did I set it out right? Did I remember what I was meant to do?

Which activity conforms more to the general goals of education for children in the elementary school today—the second example of unthinking recall of instructions, or the first emphasising creative problem-solving and sense making based on number relations?

However, the real argument is not between mental and written computation. The real argument is about what the main goal of arithmetic in the primary school should be. A traditional response was: speed and accuracy in written computation. In this book we suggest that the appropriate response today is *the acquisition of number sense*.

Number sense refers to:

A person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations.... It is an important underlying theme as the learner chooses, develops and uses computational methods. (McIntosh, Reys & Reys, 1995, p. 3)

One story, a true one, sticks in my mind as exemplifying the difference between skill in written computation and number sense. I had occasion some years ago to buy two identical articles, marked as costing \$2.50 each, but labelled 'half marked price'. I took them to the counter where the assistant picked up the first article, wrote \$2.50, performed a written division by 2, and obtained the result \$1.25. She then picked up the other, identical article, wrote \$2.50, performed a second written division of the same numbers by two, and again obtained \$1.25. She then wrote \$1.25 twice, one below the other, drew a line, performed a written addition, and obtained \$2.50. She handed them to me and said, with no flicker of any expression: "That will be \$2.50 please". I cannot fault her written computation skills. But I do think that she showed no glimmer of number sense.

Faced with a problem situation, people with good number sense see that a quantitative solution is required and are not deterred by this. They ask themselves, *What kind of answer is needed—approximate or exact?* If approximate, then an estimate is needed. If exact, then the next task is to choose an appropriate method of solution: depending on the complexity of the calculation required, and having a reasonable proficiency in a variety of methods, people with good number sense choose between computer, calculator, written or mental computation, produce an answer, and check this answer for reliability by comparing it with an estimate.

If we are to take this emphasis seriously, then we must pay attention to all the loops and make sure our actions with children all move in the direction of the central aim. If emphasis on one aspect conflicts with this, then we must rethink this emphasis. It is not efficient to concentrate all the instructional attention on one method of calculation (even if that in itself were successfully achieved), and expect that all the other aspects of number sense will somehow fall into place.

Apart from the introduction of calculators, none of this is less than fifty years old, as theory. As some indication that current rule-and-written-based teaching is less than successful in instilling number sense in our students, Table 1 presents some results of testing Professors Robert and Barbara Reys of the University of Missouri-Columbia and myself and other colleagues carried out in 1993 in Australia, the United States, Sweden, Japan and Taiwan (McIntosh, Reys, Reys, Bana, & Farrell, 1997).

Table 1: International student performance on selected number sense items

How many decimals are there between 1.52 and 1.53?	<i>23% of Year 9 students said None (AUS)</i>
How many fractions are there between $\frac{2}{5}$ and $\frac{3}{5}$?	<i>29% of Year 7 students said None (AUS)</i>
Name a fraction between $\frac{1}{2}$ and 1, other than $\frac{3}{4}$?	<i>44% of Year 7 could not (AUS)</i>
10% of the boys and 10% of the girls in a school smoke. What percent of the students in the school smoke?	<i>64% of Year 8 said 20% (SWE)</i>
About how many days have you lived? 500, 5000, 50 000, 500 000 or 5 000 000	<i>7% of Year 8 students chose 5 million (SWE)</i>
Calculate $\frac{12}{13} + \frac{7}{8}$	<i>63% correctly calculated, but only 38% correctly estimated that the result was nearest to 2 [given the choices 1, 2, 19 and 21] (TAIW).</i>
Write >, = or < to make this a true statement: $456 \div 8$ _____ $456 \times \frac{1}{8}$.	<i>52% of Year 9 chose < or > (AUS)</i>
Calculate $\frac{1}{10} \times 45$	<i>60% of Year 9 would choose to do mentally (AUS)</i>
Calculate 0.1×45	<i>45% would make a similar choice to do mentally (AUS)</i>

There is nothing in the nature of mathematics, in educational theory, in psychology or in tradition that says that we have to continue to concentrate on formal written algorithms. National and State guidelines have moved, or are moving strongly in this direction. We have only ourselves, and our colleagues to convince. We hope that what follows will help you to make your own informed decision about what is appropriate.

References

- Bidwell, J. & Clason, R. (1970) (Eds.). *Readings in the history of mathematics education*. Washington, DC: National Council of Teachers of Mathematics.
- Biggs, J. (1967). *Mathematics and the conditions of learning*. Slough: National Foundation for Educational research.
- Brownell, W. (1935). Psychological considerations in the learning and the teaching of arithmetic. In *The teaching of arithmetic: The tenth yearbook of the National Council of Teachers of Mathematics* (pp. 1-31). New York: Teachers College, Columbia University.
- Brownell, W. & Moser, H. (1949). *Meaningful V Mechanical learning: A study in Grade 3 subtraction*. Duke University Studies in Education, Number 8. Durham, NY: Duke University Press.
- Cockcroft, W. (1982). *Mathematics counts. Report of the Committee of Inquiry into the teaching of mathematics in schools under the Chairmanship of Dr W.H. Cockcroft*. London: Her Majesty's Stationary Office.
- Ewbank, W. (1977). Mental arithmetic a neglected topic? *Mathematics in School*, 6(5), 28-31.
- Hope, J. (1986). Mental calculation: Anachronism or basic skill? In H. Schoen & M. Zweng (Eds.), *Estimation and mental computation* (pp. 45-55). Reston, VA: NCTM.
- Hope, J. (1987). A case study of a highly skilled mental calculator. *Journal for Research in Mathematics Education* 18(5), 331-342.
- Jones, P. (1988). Mental mathematics moves ahead. *Mathematics in School*, 88, 42-44.
- Maier, E. (1980). Folk mathematics. *Mathematics Teaching*, 93, 21-23.
- McIntosh, A., Northcote, M. & Sparrow, L. (1999). Adults' non-occupational mathematics: Wandt and Brown revisited. In J. M. Truran & K. M. Truran (Eds.), *Making the Difference*. Proceedings of the 22nd Annual Conference of the Mathematics Education Research Group of Australasia. Adelaide: MERGA.
- McIntosh, A., Reys, B.J. & Reys, R.E. (1992). A proposed framework for examining number sense. *For the Learning of Mathematics*, 12(3), 2-8.
- McIntosh, A., Reys, B., Reys, R., Bana, J., & Farrell, B. (1997). Number sense in school mathematics: Student performance in four countries. Perth: MASTEC, Edith Cowan University.
- Nunes, T., Carraher, D. & Schlieman, A. (1987). Written and oral mathematics. *Journal for Research in Mathematics Education*. 18(2), 83-97.
- Plunkett, S. (1979). Decomposition and all that rot. *Mathematics in School*. 8(3), 2-7.
- Reys, R.E. (1984). Mental computation and estimation: Past, present and future. *Elementary School Journal*, 84(5), 546-557.
- Stigler, J. & Perry, M. (1990) Mathematics learning in Japanese, Chinese and American classrooms. In L. Steffe, P. Nesher, P. Cobb, G. Goldin & B. Greer (Eds.), *Cultural psychology: Essays on comparative human development* (pp. 328-353) Cambridge: Cambridge University Press.

- Trafton, P. (1986). Computation—It's time for a change! *Arithmetic Teacher*, 34(3), 2.
- Wandt, E. & Brown, G. (1957). Non-occupational uses of mathematics: Mental and written—approximate and exact. *Arithmetic Teacher* 4(4), 151-154.

Part 1

Young Children's Number Concepts

Introduction

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Beyond Written Computation describes recent changes in approach to the development of children's ability to calculate and handle quantitative situations. But as any teacher knows, there is a great deal of prior learning about number that precedes even the earliest encounters with calculating. Moreover, knowledge and understanding of number deepens and spreads throughout a child's education.

Part 1 provides three perspectives on the development of young children's ideas of numbers and operations, and in doing so emphasises three central themes of the book: the importance of conceptual understanding, the central role of the teacher, and the exploration of critical incidents.

Mulligan provides an Australian overview of key aspects of early number learning critical to the development of number sense and computation, drawing on experience of a major project originating from New South Wales. She exemplifies the value of a research-based approach to the curriculum, describes and illustrates four key aspects of early number learning, and draws on encounters with individual students to illuminate the narrative. Amongst other important relationships described is that between calculating and increasingly sophisticated methods of counting.

Askew, Bibby and Brown provide a British perspective on early number work, drawing on experience of a large-scale British research project aimed at improving standards of early numeracy, particularly of low achievers. They highlight three key issues: the role of mental computation, the place of practical work and 'real' contexts, and the critical interface between assessment and teaching. In particular they re-assess and re-emphasise the central role of the teacher.

Whitenack, Knipping, Novinger and Underwood pick up one of Mulligan's four key aspects of early number learning—place value; and also on Askew and his colleagues' focus on the teacher. They illuminate and illustrate both aspects by a close focus on two critical classroom incidents in a project that draws on experience in two major studies in the United States. They illustrate the importance of teachers' knowledge and understanding of children's conceptual development and the role of sensitive and open classroom discussion, and clarify the constant and detailed decision making involved in teaching.

Key Aspects of Early Number Learning

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The teaching and learning of early numeracy in the 21st century requires educators to take a more holistic view of the complexities of young children's mathematics learning. The increasing diversity of young children's life experiences that give rise to fundamental aspects of mathematics has changed the way they come to understand their world quantitatively. Widespread and early exposure to information technologies including multimedia has influenced the way children acquire and represent informal notions of mathematics (Diezmann & Yelland, 2000). As well, our expectations of the capabilities of young children have been raised in response to their growing capacity to develop and apply mathematical concepts much earlier than previously expected.

Early mathematical literacy is now a priority at international level with government projects targeted at improving numeracy in schools generally (Department of Education, Training and Youth Affairs, 2000; Brown, 2000; National Council of Teachers of Mathematics, 2000; Perry, 2000). If our common goal is to develop young children's 'number sense', this means that we may need to place a major emphasis on those mathematical processes that promote the development of a flexible range of numerical strategies. The development of efficient mental calculation strategies, based on informal methods, has emerged as a new priority for early numeracy. This is in sharp contrast to traditional emphasis on written algorithmic procedures.

This chapter describes some key aspects of early number learning critical to the development of number sense and mental computation, and mathematics generally: *counting and estimation, grouping and partitioning, base ten, and arithmetical strategies*. These aspects can be seen as interrelated parts of a complex framework of number ideas and relationships. Several classroom examples will be used to illustrate these aspects and how children can use a range of strategies in interrelated ways. A Learning Framework in Number (LFIN) will exemplify a structured way of mapping the development of children's early numerical strategies (NSW Department of Education & Training, 1998). First, some background research will provide an overview to current perspectives on early numeracy in the 21st century.

Background

In past decades attention has been focused on how young individuals develop specific concepts or skills such as counting, addition and subtraction, and how teaching might foster number learning. Traditionally, early number learning was perceived as 'pre-number', viewed as a set of Piagetian-based hierarchical skills such as sorting and

classification, conservation, cardinal and ordinal number. These fundamental aspects of early number learning remain important but current direction has shifted to the child's development of a rich repertoire of arithmetical concepts and strategies (Tang & Ginsburg, 1999). The rate and methods by which the child develops these aspects is considered to vary considerably for individuals.

More recent emphasis has been placed on the teacher's role in the learning process and how social interactions impact on children's mathematical learning overall (Cobb & McClain, 1999). Children are seen as individuals within the classroom as a collaborative learning environment. The role of the teacher is to identify and promote those strategies that contribute explicitly to the development of children's increasingly sophisticated mathematical strategies. Children are being encouraged to develop a range of strategies to solve mathematical problems, and to reflect upon, justify and explain their own strategies and the strategies of others.

Australian government initiatives in early numeracy such as the NSW 'Count Me In Too' project (CMIT) and the Victorian 'Early Number Research Program' (ENRP) have assisted teachers in developing their pedagogical knowledge in order to support students' development (Clarke, Sullivan, Cheesman, & Clarke, 2000; Gould, 2000). Frameworks for early number learning have been developed to provide a basis for teachers to promote the development of numerical strategies (Wright, 1998).

Research on Early Numeracy

Extensive research on the development of early number knowledge has provided a more coherent picture of the development from informal to formal numerical ideas (Mulligan & Mitchelmore, 1996; Steffe, Cobb, & Richards, 1998; Wright, Martland, & Stafford, 2000). The early acquisition of counting, partitioning and grouping strategies, base ten and arithmetical knowledge has been highlighted as critical to mathematical learning. Studies focused on children's solutions to word problems have identified the development of sound problem-solving strategies and the importance of modelling and representation in this process (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Clark & Kamii, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1997).

The use of imagery has also been linked to young children's conceptual development of number and the way they represent numerical ideas (English, 2004; Gray & Pitta, 1996; Mulligan, Mitchelmore, Outhred, & Russell, 1997; Thomas & Mulligan, 1995; Thomas, Mulligan, & Goldin, 2002). Children's images, whether they are shown as primitive pictures, icons or more abstract notations, can determine how they represent number sequences, for example, and the way they might structure mathematical situations. There has been much attention on generating and assessing pupil work samples of mathematical ideas including their drawings and diagrams of solution strategies, and explanations of how they use numerical ideas (Board of Studies NSW, 2002; Diezmann & Yelland, 2000).

Given the extensive work on early numeracy, some common findings are summarised as follows:

1. Children's informal and intuitive numerical images, ideas, explanations and recordings form a very important basis in developing numerical concepts (Bobis, 1996; Gifford, 1995; Hughes, 1986; Mulligan & Mitchelmore, 1996);

2. Many children begin school with a large repertoire of numerical strategies such as counting, grouping, partitioning, and computational skills, which have been developed earlier than traditionally expected (Carpenter et al., 1993; Clark & Kamii, 1996; Macmillan, 1995; Wright, Mulligan, & Gould, 2000; Young-Loveridge, 1997);

3. Children need to develop and recognise underlying structures in order to understand how the number system is organised and ordered by grouping in tens etc. (Hiebert & Wearne, 1992; McClain & Bowers, 2000; Thomas & Mulligan, 1999; Whitenack, Knipping, Novinger, & Underwood, *this volume*);

4. Children can develop counting and arithmetical strategies by devising their own problems and solving simple problems related to the four operations; they can discuss, explain and record their thinking using numerical representations that approximate conventional notations (Carpenter et al., 1993; English, 2004; Mulligan & Mitchelmore, 1996);

5. Children need to develop number sense through flexibility in the way they use mental strategies (McIntosh, 1998; Trafton, 1999);

6. Children can develop increasingly sophisticated counting and arithmetical strategies by challenging them to think abstractly rather than relying on concrete or visual models (Boulton-Lewis, 1998; Wright, 1998); and

7. Children can use calculators effectively to promote their numerical concepts and skills and problem-solving strategies (Groves & Stacey, 1998).

Research on the development of early number knowledge in Australia has been extensive (Doig, Groves & Mulligan, 2004; McIntosh & Dole, 2000; Wright, Mulligan, Bobis, & Stewart, 1996). This has highlighted the importance of young children's developing number 'strategies' and 'number sense'. Number strategies refer to particular methods or skills that the child develops either intuitively or from instructional experiences such as counting-on, counting back, sharing, and grouping. Number sense refers to the child's general understanding and use of numerical concepts and strategies as well as the ability to apply these in flexible ways. For young children this may involve applying a known strategy to a new situation such as using doubles, combining strategies such as doubling and subtracting one, making sensible estimates and thinking about the reasonableness of an answer.

Key Aspects of Early Number Learning: Seeing Structure and Interrelationships

In the following section some classroom examples of children's use of key aspects of number learning are shown: *counting and estimation, grouping and partitioning, base ten and arithmetical strategies*. No single aspect alone can determine the growth and development of number knowledge. Children's pre-calculation and mental computation strategies are based on developing these aspects with increasing levels of sophistication. Critical to this development is the growth of abstract thinking that promotes effective mental computation strategies.

It is also crucial to look at the way children establish, or fail to establish, relationships between one aspect and another. Ideally this requires that children find similarities and differences between one numerical aspect and another, such as using a counting sequence of multiples to form equal groups. Seeing these connections will

enable children to develop a flexibility that enables them to develop a coherent range of arithmetical (computational) strategies and where they can move between one strategy and another. Existing strategies can be extended to solve a new, related problem; or a new strategy may be constructed that builds upon prior strategies. Eventually, children can develop an '*astuteness*' that one strategy may be more effective, or more efficient, than another when calculating mentally or solving a simple problem. We also need to consider how individuals vary in developing their own methods for finding patterns and relationships between these key aspects and how this might influence their operating with larger numbers later.

Underpinning the effective development of key aspects of number learning and corresponding mental strategies is the child's ability to see '*structure*' in numerical processes and representations. Structure can be identified in a variety of ways such as by finding patterns of five on an array of twenty-five items rather than seeing twenty-five individual items. Some children impose their own structure on mathematical situations and this may enhance or impede effective solution strategies.

Recognising and creating patterns is also fundamental to developing underlying structure. The following examples show *estimating* and *subitising* where children discriminate between dot patterns to identify the quantities. Subitising is the process of immediately recognising how many items are in a small group. A series of dot-pattern cards (1 to 10) were flashed and the child asked to give an estimate of the number of dots. Then they were asked to match the cards that had the same number of dots and explain why one card was easier to quantify than another. Similarly other patterns of the numerals 5 to 10 were explored. Figure 1 shows Emma's ability to form different structures of eight (Bobis, Mulligan, & Lowrie, 2003, p. 131).

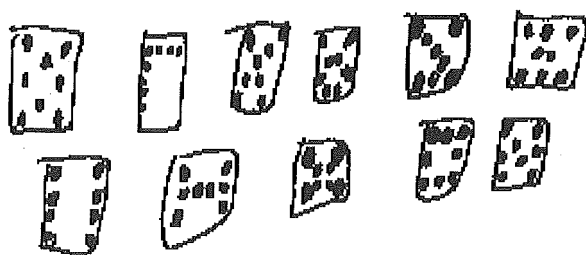


Figure 1: Similarities and differences between patterns of 8 (Emma, Kindergarten)

Identifying the quantities on dot cards can be extended to include many aspects of number learning by requiring children to match the patterns with objects, draw the patterns from memory, record the numerals and number names, order the cards numerically and generate new patterns such as odds and evens. Children can also be encouraged to create their own spatial arrangements. Bobis (1996) describes how a teaching program using dot patterns enabled students to successfully recognise and construct number patterns and basic combinations in the first year of schooling.

The development of number sense overarches the child's general understanding of number concepts and strategies. It is important that the child can gain a *flexibility* or a 'feeling' for how the number system works. Samantha, aged 5 years 2 months, gives a good example of her mental picture of the number sequence from 1 to 20 as shown in Figure 2.

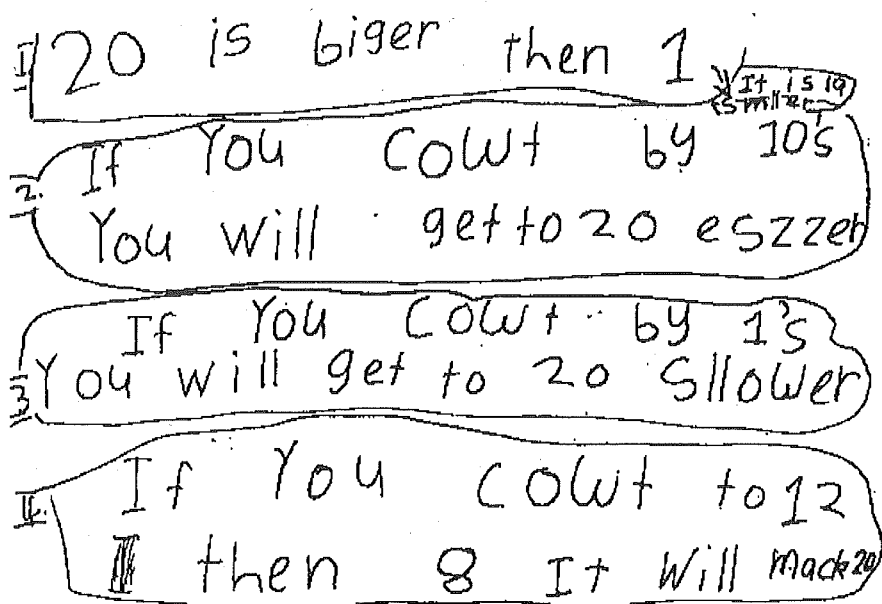


Figure 2: Samantha counting from 1 to 20

Developing Mental Computation Strategies

Three examples, showing a range of mental computation strategies are drawn from classroom studies of children in the first two years of schooling.

Example 1: Counting, addition and multiplication strategies

In response to the task, "Start with two and the answer is ten", Kindergarten children demonstrated wide differences in the mental computation strategies they used to solve and explain relationships between the numbers two and ten.

Excerpt 1 (Samantha)

You start with two and you go 3, 4, 5, 6, 7, 8, 9, 10 (shows fingers for each count) and you have eight more.

Excerpt 2 (Tran)

Four, six, eight, ten. Eight more makes ten (shows fingers for each count).

Excerpt 3 (Paul)

Two and three makes five, and five more makes ten. I doubled five. [Five] and the three makes eight altogether.

Excerpt 4 (William)

You could break the ten into five bits so there's two in each, so five twos make ten.

Excerpt 5 (Anna) "Ten is five twos, so ten is five times bigger.

These excerpts show increasingly sophisticated strategies that can be distinguished in a number of ways. Excerpt 1 shows that Samantha used a count-by-ones strategy and records the process using her fingers. This is often referred to as perceptual counting (Wright, 1998). Tran shows an advance on this strategy by using the multiple pattern of two but is still at a perceptual level. There is a marked difference in the strategies used by Paul who uses a partitioning of five and doubles to arrive at the combination of three and five. This flexibility is a basis for developing more complex mental computation strategies. Excerpts 4 and 5 are distinguished by the children's ability to use multiplicative ideas rather than counting, or additive strategies where multiplication is used as an operation—"ten is five times bigger". If young children are developing these more advanced strategies in simple situations the transfer of these strategies to situations involving larger quantities can take place.

Example 2: Base ten and equal grouping

The numeration system is often described as a base ten number system and the structure of the system needs to be understood by children in order for them to extend the system to larger whole numbers and to include decimal numbers (Hiebert & Wearne, 1992). One of the difficulties children experience is that they learn the place value of units (or ones), tens, hundreds and thousands without seeing the pattern of tens. They need to see that the system of numeration is based on the use of ten as a unit (ten ones makes one ten) and each place value is created by multiplying by ten. In the next chapter of this volume, Whitenack et al. show how children's flexible thinking about collections of tens and ones should precede or occur concurrently with teaching addition and subtraction of two-digit numbers.

The idea of forming equal groups, particularly groups of ten, is an advance on the early counting and mental calculation strategies exemplified above. Equal grouping requires the child to form groups of equal size and use these groups as units. Children's mental computation strategies for estimating or calculating the number of items in a group can distinguish important differences between using unitary counting and additive or multiplicative ideas. In a longitudinal study of early number concepts (Mulligan et al., 1997), 120 Year 2 students were shown a random collection of counters with each example of increasingly difficult number size, (i) 10 items (ii) 20 items or (iii) 100 items. The students were asked:

You don't need to work out how many counters there are but can you show me an easy way of working out how many counters there are very quickly? Are there any other ways that you could group the items?

These responses to the task with 100 counters revealed a wide range of strategies with 24% of students using unitary (count by ones) strategies. For example, Amanda made several attempts to count the items by ones but lost track of her counting and the items each time. Jason used a multiple counting pattern (2, 4, 6, ...) by placing the counters in equal groups of 2 and counting by twos to 100. Others used variations of the pattern of twos by doubling and redoubling. More than 50% of students used quinary-based strategies showing some structural characteristics where the formation of equal-sized groups, rows or arrays of five counters allowed the students to calculate the total easily if they needed to.

One student used his hand as an informal unit of measure in such a way that covering eight counters with one and then using the hand as a unit. "There's about 12 hands worth of counters and eight in one hand, so I would count up eight, twelve times". Two of the most sophisticated responses showed how students used spatial patterns to form groups.

I made an empty array by making a row of ten and a column of ten in an L Shape and then you could fill it in with counters. I'd know if there were 100 because it would be ten tens.

Another student suggested that the counters could be placed on grid paper in rows of 10 (with squares of approximate size to the counters) and counting the number of rows using repeated addition. The development of increasingly sophisticated grouping and calculation strategies can be enhanced when the teacher is aware of the level of strategy use by the student. The modelling of equal groups and the associated counting patterns can assist students in moving beyond unitary and simplistic counting methods to see the structure of equal groups and determining the most efficient group size—for example, when groups of ten are more appropriate than groups of two.

Example 3: Using interrelationships

Base ten strategies can be used to assist the child to combine and extend existing grouping and partitioning strategies. The following example shows how Timothy, aged 5 years, calculates four groups of fifteen using mental strategies based on grouping, partitioning and combining tens and fives.

Fifteen kids in each group and 4 groups, I can do it in tens ... 10, 20, 30, 40 (flashes ten fingers simultaneously with count) makes forty, and the fives, 5, 10, 15, 20 (flashes five fingers simultaneously with count), forty and twenty, that's 40, 50, 60 ... that's 60 kids all together.

Susie, aged six years, uses an alternative but equally sophisticated strategy, based on partitioning tens and fives and doubling, which is shown in the following example:

15 and 15, that's 10 and 10 makes 20, and 5 and 5 makes 10 ... 20 and 10 makes 30 ... so for 4 groups, it would be 30 and 30 ... 30, 40, 50, 60 (tapping ten fingers on table simultaneously with count).

These examples show how children combine their counting and base ten strategies with the notion of equal grouping. These children can reorganise the quantities according to the structure that is most efficient for them—that is, they can work in fives and tens. Tim partitions 15 into 10 and 5; Susie collects the tens and fives separately and then combines. We would not normally expect children of this age to represent and calculate 4×15 . The promotion of multiple counting and base ten strategies in the first year of schooling enabled these children to apply their knowledge of counting and partitioning effectively.

A Learning Framework in Number

Since 1996 the Count Me in Too Project (CMIT) has been implemented in over 1400 schools in NSW, as well as in Tasmania, ACT and New Zealand. The project is a school-based professional development initiative of the NSW Department of Education and Training focused on early numeracy. It incorporates a Learning Framework in Number (LFIN), based on the work of Steffe (1992), Wright (1998) and Mulligan

(Mulligan & Wright, 2000), which exemplifies five key aspects of arithmetical development of young children:

- arithmetical strategies such as counting-on,
- number word sequences such as counting forwards and backwards,
- base ten strategies such as using tens and units simultaneously,
- arithmetical procedures such as combining and partitioning and patterning, and
- early multiplication and division where equal grouping is developed.

Primarily, children are assessed and engaged in numeracy tasks that challenge their current level of strategy development. A key feature of the LFIN is the development from a perceptual level where the child relies on counting individual items to a level where the child can use advanced counting strategies abstractly. Each successive level shows the child's cognitive advances as well as new conceptual understandings. Once basic counting processes are in place the key aspects, grouping and partitioning, base ten and equal grouping form the core of the framework.

As the focus of the project is the advancement of students' mathematical solution strategies, assessment of this growth attempts to show the most advanced strategies a student can elicit. The comparison of the rates of change of strategy between the initial and final assessments suggests that the Count Me In Too project has progressed students' development of solution strategies from less efficient to more efficient, ahead of expectations (Gould, 2000). This growth can be seen primarily in terms of the students' ability to develop more sophisticated counting and arithmetical strategies based on a teaching program incorporating efficient counting and patterning. Students making considerable progress are developing a more effective range of strategies and stronger number sense.

Implications for Formal Written Algorithms

Effective mental computational strategies, routines and algorithms in the primary years can be traced to the development of children's informal and intuitive strategies and semi-informal methods (Gravemeijer, van Galen, Boswinkel, & van den Heuvel-Panhuizen, *this volume*). Problem situations can be modelled to support students' situation specific solution strategies such as the empty number line. "Models help students structure their way of working and this lays the basis for flexible routines later". Children's mental strategies should be based then, on the strategies that they are familiar with. As shown in the examples in this chapter, children can develop a range of rich strategies for effective mental computation by flexible use, and extension of strategies they have already developed.

Over thirty years ago, eminent British mathematics educator, Edith Biggs, called for the review of teaching formal algorithmic processes. "This then is a crucial test of readiness for practice in written computation with tens and units: the ability to add two 2-digit numbers mentally by an efficient method" (Biggs, quoted in Ewbank 1977, p. 29). Her message requires us to revisit the purpose of learning algorithms. We need to think beyond the goal of learning written algorithms and focus on critical mental computation strategies as a goal in themselves. The acquisition of efficient mental computation strategies can eliminate, or reduce, the need for formal written algorithms.

Some mathematics educators are now questioning whether formal written algorithmic procedures may even impede the development of effective mental computation skills (Boulton-Lewis, 1996, Kamii & Dominick, 1998).

If children can use efficient mental strategies to add and subtract pairs of two digit numbers, and multiply and divide two-digit by one-digit numbers then it may be more appropriate and efficient to use a calculator beyond this point. However, despite the encouragement to use calculators in this capacity, teachers have not generally replaced traditional algorithmic procedures with their use. Even if calculators are not encouraged, effective mental computation strategies can still replace algorithmic procedures at the early level (see chapters by Stacey, and by Groves, *this volume*).

Conclusions

In this chapter, children's development of increasingly sophisticated mental computation strategies have been described in terms of key aspects of developing number knowledge: counting and estimation, grouping and partitioning, base ten and other arithmetical strategies. The fundamental aspects of early number learning may appear not to have changed from traditional aspects but they are portrayed in a more complex and integrated way which encourage the child to construct a rich variety of arithmetical strategies, number concepts and relationships. Traditionally, young children in the first year of schooling have engaged in a sequence of learning activities that match curriculum prescriptions such as rote counting to ten, and learning about numerals in a lock-step fashion. The assessment of young children's existing strategies makes it apparent that many are entering their first year of schooling with abstract counting skills, some basic number facts and mental computation strategies, and some base ten and equal grouping skills. This is not necessarily in keeping with traditional curriculum guidelines or teaching programs.

We need to re-direct our attention to helping children develop the increasingly effective and fundamental strategies they need in order to acquire secure arithmetical understanding, knowledge and skills. The mathematics curriculum in the early years of schooling has already begun to remove boundaries in terms of content and grade level restrictions. An outcomes-based curriculum (Board of Studies NSW, 2002; Ministry of Education, Victoria, 1997) means that teachers can promote students to work at their own level and move beyond traditional syllabus expectations.

Assessing children's critical mental computation skills will create more opportunities for developing numerical concepts and strategies traditionally delayed until later years of schooling. More importantly, children can develop a repertoire of rich and effective mental strategies that can be used flexibly and reliably in a range of mathematical situations. The development of mental computation skills will re-focus attention on children's potential to use effective mental processes, rather than a conventional belief that modelling with concrete materials and acquiring procedural written algorithmic skills will ensure basic numeracy.

References

- Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics*, 20, 367-385.
- Board of Studies, NSW. (2002). *Mathematics K-6 syllabus 2002*. Sydney: Board of Studies, NSW.
- Bobis, J. (1996). Visualisation and the development of number sense with Kindergarten children. In J. T. Mulligan & M.C. Mitchelmore (Eds.), *Children's number learning* (pp.17-35). Adelaide: Australian Association of Mathematics Teachers/Mathematics Education Research Group of Australasia.
- Bobis, J., Mulligan, J., Lowrie, T. (2003). *Mathematics for children: challenging children to think mathematically*. 2nd Edition. Sydney: Pearson.
- Boulton-Lewis, G. (1998). Children's strategy use and interpretations of mathematical representations. *Journal of Mathematical Behavior*, 17, 219-239.
- Brown, M. (2000). What kinds of teaching and what other factors accelerate primary pupils' progress in acquiring numeracy? Paper presented to the ACER Annual Conference 2000, *Improving numeracy learning: What does research tell us?* (pp. 1-3). Melbourne: Australian Council for Educational Research.
- Carpenter, T.P., Ansell, E., Franke, K.L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of Kindergarten children's problem solving processes. *Journal for Research in Mathematics Education*, 24, 428-441.
- Clark, F.B. & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education*, 27, 41-51.
- Clarke, D., Sullivan, P. Cheesman, J. & Clarke, B. (2000). The early numeracy research project: Developing a framework for describing early numeracy learning. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000*. Proceedings of the 23rd Annual Conference of the Mathematics Education Research Group of Australasia, (pp. 180-188). Perth: MERGA.
- Cobb, P. & McClain, K., (1999). Supporting teachers' learning in social and instructional contexts. *Proceedings of the 1999 International Conference on Mathematics Teacher Education*. Taiwan: National Taiwan Normal University.
- Department of Education, Training and Youth Affairs (2000). *Numeracy, a priority for all: Challenges for Australian schools*. Canberra: DETYA.
- Diezmann, C.M. & Yelland, N.J. (2000). Developing mathematical literacy in the early childhood years. In N.J. Yelland (Ed.), *Innovations in practice: Promoting meaningful learning for early childhood professionals* (pp. 47-58). Washington: National Association for the Education of Young Children.
- Doig, B., Groves, S. & Mulligan, J. (2004). Number learning in the primary years. In B. Perry, G. Anthony & C. Diezmann (Eds.) *Research in mathematics education in Australasia 2000-2003*, (pp. 245-265). Flaxton, Queensland: MERGA.
- English, L. (2004). *Mathematical and analogical reasoning of young learners*. Mahwah, NJ: Lawrence Erlbaum.
- Ewbank, W.A. (1977). Mental arithmetic a neglected topic? *Mathematics in School*, 6(5), 28-31.

- Gifford, S. (1995). Number in early childhood. *Early Child Development and Care*, 109, 95-119.
- Gould, P. (2000). Count me in too: Creating a choir in the swamp. Paper presented to the ACER Annual Conference 2000, *Improving numeracy learning: What does research tell us?* (pp. 23- 26). Melbourne: Australian Council for Educational Research.
- Gray, E., & Pitta, D. (1996). Number Processing: Qualitative differences in thinking and the role of imagery. In L. Puig and A. Guitierrez (Eds.), *Proceedings of the 20th Annual Conference of the International Group for the Psychology of Mathematics Education: Vol. 4* (pp. 155-162). Valencia, Spain: Program Committee.
- Groves, S. & Stacey, K. (1998). Calculators in primary mathematics: Exploring number before teaching algorithms. In L.J. Morrow & M.J. Kenney (Eds.) *The teaching and learning of algorithms in school mathematics* (pp.120-130). Reston VA: National Council of Teachers of Mathematics.
- Hiebert, J. & Wearne, D. (1992). Links between teaching and learning place value with understanding in first grade. *Journal for Research in Mathematics Education*, 23, 98-122.
- Hughes, M. (1986). *Children and number*. Oxford: Basil Blackwell.
- Kamii, C. & Dominick, A. (1998). The harmful effects of algorithms in grades 1-4. In L.J. Morrow & M.J. Kenney (Eds.) *The teaching and learning of algorithms in school mathematics*. (pp.130-141). Reston, VA: National Council of Teachers of Mathematics.
- Kouba, V.L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education*, 20, 147-158.
- Macmillan, A. (1995). Children thinking mathematically: Beyond authoritative identities. *Mathematics Education Research Journal*, 7(2), 111-132.
- McClain, K. & Bowers, J. (2000). Supporting preservice teachers' understanding of place value and multidigit addition and subtraction. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Annual Conference of the International Group for the Psychology of Mathematics Education: Vol.3* (pp. 279-287). University of Hiroshima, Hiroshima: Program Committee.
- McIntosh, A. (1998). Teaching mental algorithms constructively. In L.J. Morrow & M.J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics*. (pp. 44-49). Reston, VA: National Council of Teachers of Mathematics.
- McIntosh, A. & Dole, S. (2000). Early arithmetical learning and teaching. In K. Owens & J. Mousley (Eds.) *Research in mathematics education in Australia 1996-1999*. Sydney: MERGA.
- Ministry of Education, Victoria, (1997). *Mathematics Curriculum Framework*. Melbourne: Ministry of Education.
- Mulligan, J.T. & Mitchelmore, M.C. (1996). Children's representations of multiplication and division problems. In J.T. Mulligan & M.C. Mitchelmore (Eds.), *Children's number learning* (pp.163-185). Adelaide: Australian Association of Mathematics Teachers Inc./ Mathematics Education Research Group of Australasia.

- Mulligan, J.T. & Mitchelmore, M.C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education*, 28, 309-331.
- Mulligan, J.T., Mitchelmore, M.C., Outhred, L. & Russell, S. (1997). Second grader's representations and conceptual understanding of number. In F. Biddulp & K. Carr (Eds.), *People in mathematics education*. Proceedings of the 20th Annual Conference of the Mathematics Education Research Group of Australasia, Vol 2, (pp. 361-369). Rotorua, NZ: MERGA.
- Mulligan, J.T. & Wright, R.J. (2000). Interview-based assessment of early multiplication and division. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Annual Conference of the International Group for the Psychology of Mathematics Education: Vol. 4* (pp. 17-25). University of Hiroshima: Program Committee.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- NSW Department of Education and Training (1998). Count me in too: Professional development package. Sydney: NSW Department of Education and Training Curriculum Directorate.
- Perry, B. (2000). Early childhood numeracy. Paper commissioned by the Australian Association of Mathematics Teachers Inc. for the Commonwealth Department of Education Training and Youth Affairs. Canberra: Commonwealth of Australia.
- Steffe, L. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4, 259-309.
- Steffe, L., Cobb, P. & Richards, J. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.
- Tang, E. & Ginsburg, H. (1999). Young children's mathematical reasoning: A psychological view. In L. Stiff & F. Curcio (Eds.), *Developing mathematical reasoning in grades K-12*. Reston, VA: National Council of Teachers of Mathematics.
- Thomas, N., & Mulligan, J.T. (1995). Dynamic imagery in children's representation of number. *Mathematics Education Research Journal*, 7, 5-25.
- Thomas, N. & Mulligan, J.T. (1999). Children's understanding of the number system. In J.M. Truran and K.M. Truran (Eds.), *Making the difference*. Proceedings of the 22nd Annual Conference of the Mathematics Education Research Group of Australasia, (pp. 477-485). Adelaide: MERGA.
- Thomas, N., Mulligan, J.T. & Goldin, G. A. (2002). Children's representations of the counting sequence 1-100: Development of new cognitive structures. *Journal of Mathematical Behavior*, 21, 117-133.
- Trafton, P. (1999). *Children learning through problems: Number sense and computational strategies*. University of Northern Iowa, Exon Project: Heinemann.
- Wright, R. (1998). An overview of a research-based framework for assessing and teaching early number. In C. Kanes, M. Goos & E. Warren (Eds.), *Teaching mathematics in new times*. Proceedings of the 21st Annual Conference of the Mathematics Education Research Group of Australasia, (pp. 701-708). Brisbane: MERGA.

- Wright, R., Mulligan, J.T., Bobis, J. & Stewart, R. (1996). Research on early number learning. In B. Atweh, K. Owens & P. Sullivan (Eds.), *Research in mathematics education in Australasia 1992-1995* (pp. 281-311). Sydney: Mathematics Education Research Group of Australasia.
- Wright, R.J., Mulligan, J.T. & Gould, P. (2000). Extending the learning framework to multiplication and division, (pp. 154-179). In R.J. Wright, J. Martland & A. Stafford (Eds.), *Early numeracy: Assessment for teaching and intervention*. London: Sage.
- Wright, R.J., Martland, J. & Stafford, A. (2000) (Eds.) *Early numeracy: Assessment for teaching and intervention*. London: Sage.
- Young-Loveridge, J. (1997). From research tool to classroom assessment device: The development of checkout/Rapua, a shopping game to assess numeracy at school entry. In F. Biddulp & K. Carr (Eds.), *People in mathematics education*. Proceedings of the 20th Annual Conference of the Mathematics Education Research Group of Australasia, Vol 2. (pp. 608-615). Rotorua, NZ: Mathematics Education Research Group of Australasia.

Developing Number Sense: The Interplay of the Individual and the Social

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Three cars, three bicycles and two buses go past the school gate? How many wheels went by?

After establishing that there would be six wheels on each bus, Tom and Sam, both 8-year-olds, quickly agreed that there would be $12 + 6 + 16$ wheels. To find the total, Tom counted on from 12 but miscounted and announced 33 as his answer. Sam, after a few moments reflection, said that it was 34. Asked to explain his method, he replied: "Well there's twelve and the ten from there (pointing to the 16) makes 22, there's another six left (from the 16), so that's 28. Two from there (the 6) makes 30 and there's four left so that's 34".

Tom and Sam are friends, live near each other, play together and have been in the same class since they started school, following the same sequence of mathematics instruction. So why is it that Sam has developed more efficient and flexible strategies than Tom? And is it possible to help Tom develop approaches more akin to Sam's? While the former question may be impossible to answer, this chapter begins to explore the latter question. In particular, issues are raised about the interplay between children's individual understandings of mathematics and the expectations that they have about what it might mean to do mathematics, expectations that may be affected by the social context of mathematics teaching.

Background: The Numeracy Recovery Program

In 1992 the School of Education, King's College London, in collaboration with two London Local Education Authorities (LEAs) set up an initiative to explore strategies for developing primary students' competence and confidence in number sense, particularly for those students who had been identified as low attainers at the age of seven (the age at which English students sit their first national test).

Observations from this first phase of the work indicate that the targeted students developed greater competencies, particularly in developing a range of mental strategies and increased confidence in tackling unfamiliar problems.

Based on these observations, a follow up project (funded by the Nuffield Foundation) trailed and systematically evaluated a model for an intervention program

aimed at improving the number sense of seven-year-olds who were considered to be low attainers. In other words, the aim of this project was to help children like Tom to develop more efficient and effective ways of working with number.

The teachers of six classes of 8-year-olds in different primary schools were partners in the project. Each teacher identified eight children from their class who they considered to be low attainers in mathematics. For practical purposes, 'low attaining' was defined as performing below the expected National Test level at age seven in mathematics but not in English. We took this definition, as we wanted to work with children who were experiencing difficulty in mathematics but who did not have special educational needs across the entire curriculum. The 48 children thus chosen experienced an intervention program developed by the project team.

The six project teachers were released one day per week for twenty weeks over two teaching terms. In the first term, the teachers focused on the use and interpretation of diagnostic interviews that would be used to track the 48 children's mathematical knowledge and strategies over the course of the project. In the mornings of the release days the teachers worked intensively with their group of eight targeted students in their own schools, usually in two sub-groups of four. In the afternoons, the teachers came together to discuss the teaching strategies being developed and to work on identifying effective intervention strategies. Research findings were used to inform these discussions.

In the second term the afternoon sessions took place at an education centre, using a room with a one-way mirror. The teachers took it in turns to work with one or two students. The other teachers watched and listened through the mirror and discussed any difficulties that the children demonstrated and what teaching strategies might help them to tackle these difficulties.

In this way the work was intended to build towards a program with aims and methods similar to the Marie Clay 'Reading Recovery Program'. Ways in which our work in mathematics was similar to the reading recovery program included recognising the centrality of the teacher in the learning of mathematics. We regarded students as being 'inducted' by the teacher (Bruner, 1986) into the culture of mathematics, based on a broad Vygotskian approach, in contrast to, say, a view of learning mathematics as an individual enterprise, based on 'discovery'.

At the start of the project, Tamara Bibby, the project officer, along with a different set of class teachers, identified similar groups of eight students in six matched control schools. Thus as well as the 48 target students, 48 'control' students took part in the project. The progress and understandings of these 48 control students was tracked using the diagnostic interview, in the same way as the 48 project students. Unlike the project students, the control students would not experience any particular teaching interventions.

Three key themes emerged from the project:

- success in developing mental strategies;
- difficulties children had in 'abstracting' mathematics from teaching activities; and
- the influence of the social context on children's expectations and learning.

Success in Developing Mental Strategies

Studies of arithmetical methods used by 7- to 12-year-olds (see for example, Steffe & Cobb, 1983; Gray, 1991) demonstrate that higher attaining students have a range of alternative strategies to draw on, based on both 'knowing by heart'—recall of some number facts (for example $5 + 5 = 10$) and 'figuring out'—deriving or deducing other number facts on the basis of the known facts (for example $5 + 6$ must be one more than $5 + 5$).

On the other hand, lower-attaining students rely mainly on counting strategies based on objects (fingers or counters) or representations of objects. These findings are backed up by other strong evidence in research that, across all years of schooling, some students do not progress far beyond developing arithmetic techniques that rely on simple addition skills, such as 'counting on' or relying on repeated addition for multiplication.

It seems that students with access to both recalled and deduced number facts make more progress because each approach supports the other:

- deducing number facts helps students commit more facts to memory, and
- recalled facts help expand the range of strategies for deriving facts.

For some lower-attaining students over-dependence on counting methods, while leading (eventually!) to a correct result, removes the need to commit number facts to memory, which in turn limits their development of deductive approaches. Tom and Sam illustrate these key differences. Sam used a mix of known facts (for example, how to add on ten) and the ability to deduce other results on the basis of the known ones. Part of his ability to do this rests on the ease with which he was prepared to break the numbers involved up into chunks—for example, splitting the 6 to 'bridge through' 28 to 30. On the other hand Tom relied on a procedural, counting-based solution approach, which on this occasion let him down.

One of the main aims of our project was to see if children could actually be taught to rely less on procedural methods and more on strategic ones. Could the Toms of this world be helped to become more like the Sams? The activities developed and the intervention strategies thus focus on helping students both commit some number facts to memory and develop strategic approaches to deducing other number facts.

The children's progress was monitored using a framework for charting understanding and a related diagnostic interview (Denvir & Bibby, 2001). The children taking part in the project and those in the control groups were assessed twice using the diagnostic interview: once near the beginning of the autumn term 1995 and again in the summer term 1996. Figure 1 below shows the mean test gains for students over this period.

Figure 1 clearly shows that the project students made greater gains than the control students in terms of the number of items correctly answered in the diagnostic assessment. This gain was statistically significant at the 0.05 level (that is, the likelihood of this difference coming about just by chance was at most 5 percent).

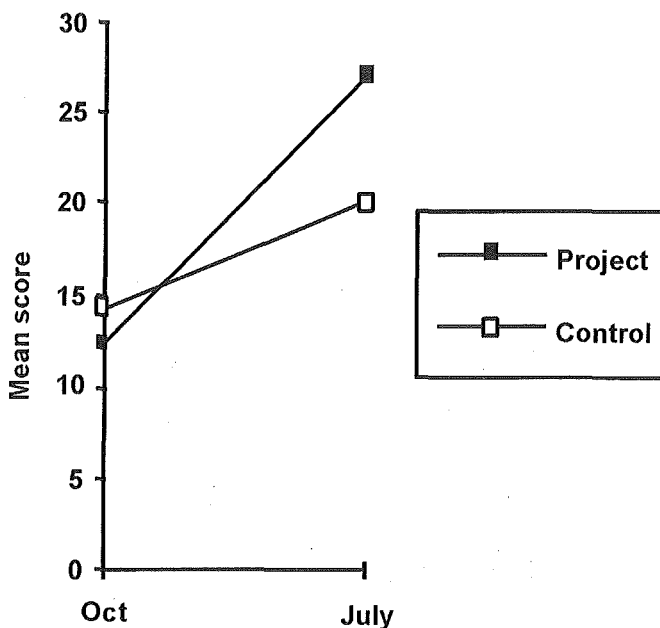


Figure 1: Mean test gains

The diagnostic interview was designed, not only to record whether or not a child could find the correct answer to a question, but also the way she or he arrived at the solution. So the research also measured changes over time in the way the children set about solving the questions in the assessment.

The methods the children chose to find a solution were coded under one of six headings, organised in increasing order of sophistication. The classification of methods is as follows.

- Not understood (NU). A child's response was recorded as not understood if she or he could not answer the question through lack of comprehension.
- Modelling (M). This indicates that the child used physical objects, including fingers, to find the answer to the question.
- Counting (Co). This means the student used a counting on or counting back method, without recourse to physical objects.
- Place value (PV). Where the children used their knowledge of place value and base-10 blocks to answer a question, they were coded PV. This category was not appropriate for all questions.
- Known fact (KF). When a student answered too rapidly to have used a calculating strategy and indicated that she or he simply knew the answer, this was coded as a known fact.
- Derived fact (DF). This coding was used to indicate that a student drew on their bank of known facts to deduce another fact.

Every item on the interview was examined for evidence of changes in strategies between the two assessments. Figure 2 shows the changes on items that any child had

not understood on the first assessment. If she or he made a minor error in calculating an answer but the method was correct, then this was coded against the method used. If she or he used an inappropriate method, or was wildly incorrect, the response was coded NU.

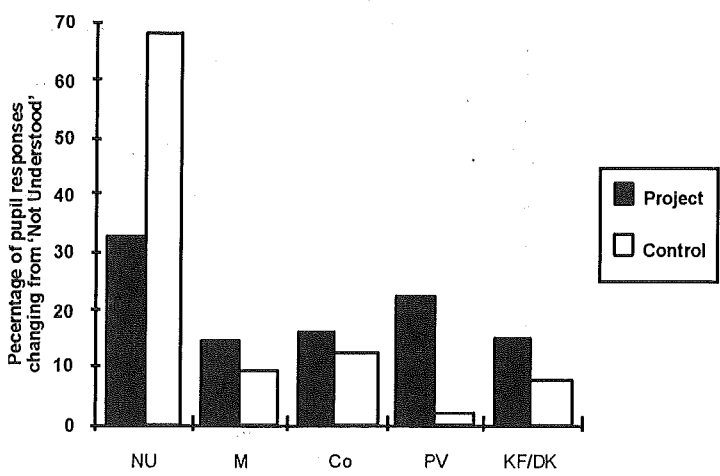


Figure 2: Changes in pupil strategies from ‘Not Understood’ – October to July

As Figure 2 shows, some items that were not understood by the children on the first assessment remained so the second time around, but the proportions for the project and control groups were very different. Nearly 70 percent of the items not understood by those in the control group in October were still not understood in July. By contrast, nearly 70 percent of the items not understood at first by those in the project groups were answered using a range of appropriate strategies. These changes are highly significant statistically ($p < 0.001$: that is, the likelihood of this difference coming about just by chance was less than 0.1 percent).

The range of strategies used by both control and project students on items not previously understood spanned modelling through to known and derived facts, but in every category the project students out-performed the control students.

Figure 3 shows the percentage changes away from a simple modelling strategy. On several items, both groups of children continued to use modelling at the later date and, in raw terms, the movement away from modelling is similar for both groups, with around 70 percent of project students and around 60 percent of control students using a different strategy. The main difference is that much of the movement on items for children in the control groups is accounted for by regression, with almost 20 percent of questions that had been answered using modelling the first time coded as not understood the second time around. The extent of regression by the children in the project groups was markedly less, at around eight percent. Again, these changes are highly significant ($p < 0.001$).

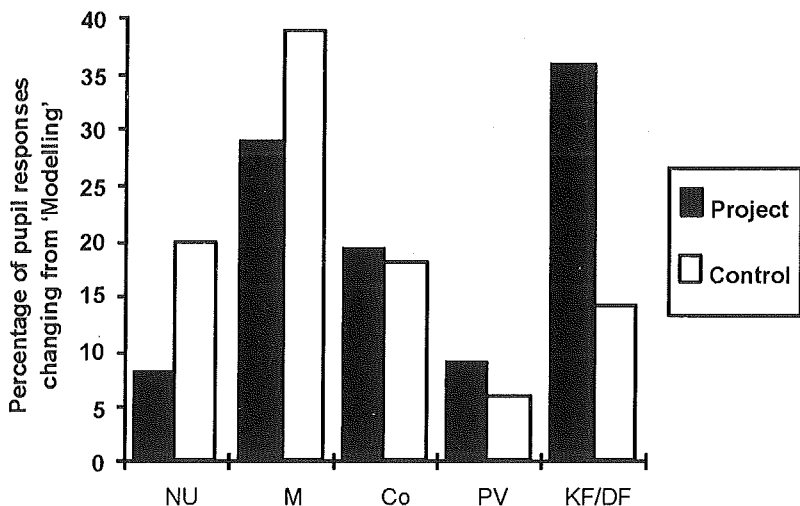


Figure 3: Changes in pupil strategies from 'Modelling' – October to July

Particularly striking is the change from using a modelling strategy to using known or derived facts. Thirty-six percent of the items that project students had originally answered using a modelling strategy were subsequently answered using a known or derived fact. The corresponding figure for control students was 16 percent.

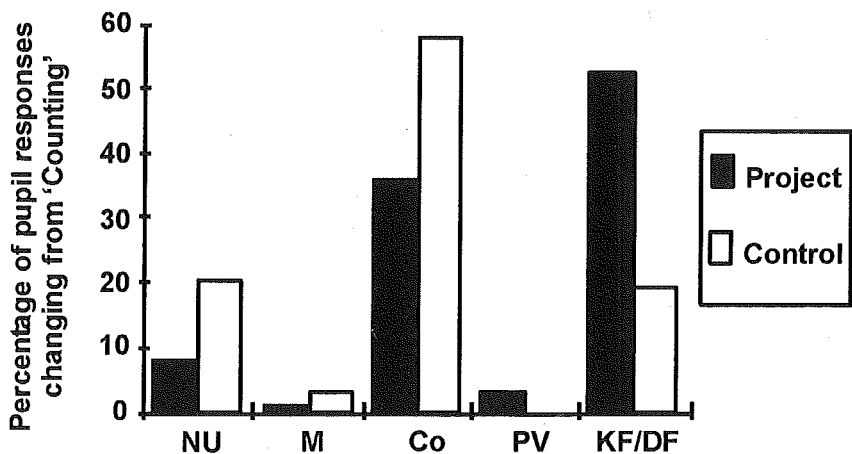


Figure 4: Changes in pupil strategies from Counting – October to July

Figure 4 shows that at the second assessment point in both groups, there was either no change or some regression on several questions answered using a counting strategy in the first assessment. The figures for the two groups are again markedly different. The children in the control groups had made no progress in strategies used in 81 percent of the items, compared to just 45 percent of those in the project. Again, these findings are highly significant statistically ($p < 0.001$). So again, the likelihood of this

difference coming about just by chance was 0.1 percent. Project students substantially out-performed control students on movement from counting strategies to the use of known and derived facts, with 51 percent of items as compared to 19 percent respectively.

All the data indicates that both in terms of the number of items correctly answered and the range of strategies used, project students significantly out-performed control students. This strongly suggests that learning to use mental strategies is not simply a matter of experience or maturation but is amenable to being taught.

Difficulties Children have in Abstracting Mathematics from Teaching Activities

While there is a long tradition of practical work in primary schools, research is also beginning to show that the use of practical materials on their own is not necessarily the best way of encouraging students to abstract mathematical concepts and develop mental strategies. For example Hughes (1986) showed that from an early age children can operate with small numbers when they are linked to objects (e.g. two elephants and two more elephants), but even after immediately being 'tuned in' to the real-world/mathematics link, they find it difficult spontaneously to put into a context numbers presented in an abstract form. Furthermore, there is little evidence to suggest that facility in particularising an abstract context to support progress in tackling a problem gets any better with age.

The Children's Mathematical Frameworks project (Hart, Johnson, Brown, Dickson, & Clarkson, 1989) confirmed this and showed more generally that the link between practical work and the move to formal symbolic mathematics is often tenuous. While teachers used practical work as a justification for formal methods, students often failed to make any firm or lasting connections between the practical and the abstract. A different look at students learning place value from tens and units blocks indicated that the blocks themselves served only as a vehicle for teacher talk—the learning came about from the way the teacher talked about and handled the blocks, rather than through the students' own discoveries (Walkerdine, 1988).

From our observations of the children in the project and their reactions to tasks set in the mirror room it became clear that one of the reasons that the children might have been low attainers was because they had come to regard mathematics as largely a practical activity. They relied on practical, counting-based methods. The example of Ben illustrates this.

Ben knew that $4 + 4 = 8$ but was unable to make the link that $4 + 5$ must be 9. Every time he was asked to do a calculation he treated it as a new situation to be worked out afresh, so rather than using his knowledge of $4 + 4$ to find the answer to $4 + 5$ he chose to use a counting method. The key to solving Ben's difficulty was to get him to make some intermediate recording. He was asked to place four counters in each of two pots and record the situation, a known fact that he could do.

Ben was then told to add another counter to one of the pots and asked if the number cards were still correct. Ben not only now knew that they were not but was able to 'correct' the recording to match the new situation. He could then do that without recalculating the total but by using his recording of the known fact. After Ben had made

this connection, his teacher reported a marked change in his attitude and approach to mathematics, demonstrating an awareness that it was something he could do in his head, rather than having to rely on external counting materials.

The main message that emerged is that while practical work and 'real' contexts can be useful, they need to be chosen carefully, and accompanied by careful dialogue with students to establish the extent of their understanding. Student success on a concrete task should not be taken as an indication of understanding the abstract. Practical and abstract may need to be presented side by side, rather than the abstract following the practical.

The Influence of the Social Context on Children's Expectations and Learning

Our analysis of the transcript of an earlier session with Ben in the mirror room suggested that part of his difficulty might have arisen from trying to do what he believed the teacher expected of him, rather than attending to the mathematics. In trying to help Ben make the connection between double 4 and $4 + 5$ the teacher set up a model of $4 + 4$, checked that Ben knew the answer, then asked him to add another counter before asking, "How many are there now?" The teacher was clearly using the word 'now' to try to link the two situations, but it appears that Ben interpreted the term differently. 'Now' seemed to suggest to him something on the lines of, "You have finished that one, now do this one". In other words, rather than encouraging Ben to make connections between the two calculations, he had interpreted this as meaning the opposite. Adding 4 and 5 was a completely new task, not something that arose out of the previous one. This is consistent with the way that most students meet arithmetic. Pages of 'sums' represent a random ordering of questions, each one to be answered independently of the one before.

Ben illustrates how low attainment is not simply a 'problem' of the child but can be a consequence of the interaction between child, activity and teacher. Other students also seemed to be doing what they thought was expected of them, rather than relying on their mathematical understanding. For example, the teachers would often ask the children to count out, say, ten cubes. Moments later, when they were asked how many cubes were there, many children would re-count them but the teachers did not discourage this. In discussion the teachers explained that they felt the students' recounting showed either that the children could not conserve number or that they could not retain the information. We encouraged the teachers to ask the children if they could remember how many there were without recounting. This the children could easily do.

In re-counting, the children seemed to be responding to what they thought their teachers expected of them, rather than doing something they needed to do. "How many are there?" for these children meant "show that you can count", which further reinforced their view of mathematics as a practical activity.

Recognising that students' interpretations of tasks and question may not fit with what was intended suggests that teachers need to spend time finding out not only what has been learned, and but also how the student interprets the task. However, research also shows that this is far easier to say than to do. Setting time aside for dialogue is itself difficult enough. Even when time is available, research suggests that it is easy to

'foreclose' on students—to jump to conclusions about a student's difficulty, either on the basis of limited information or by drawing on past experience (Bennett, Desforjes, Cockburn, & Wilkinson, 1984).

However it is also clear that teachers can improve their diagnostic and remediation skills. Crooks (1988) showed that teachers trained in diagnosis knew more about the processes that individual students used to solve problems, and their students did better in number knowledge, understanding, problem solving, and confidence. A control group tended only to explain problem-solving processes to students or just observe their students' solutions.

The Model of Teaching and Intervention

It was not our main intention to develop models for working with students on a one-to-one basis, but a pattern of working emerged that appeared to be particularly effective. The 15 minutes or so that the teachers spent working individually with students were split into four sections, as follows:

Practising counting skills (2–3 minutes)

The children would work on counting on in 2s, 5s or 10s forwards and backwards from different starting numbers. They would also work on subitising skills (recognising the number of objects in small collections without counting).

Revising individual known facts (2 minutes)

The teachers kept envelopes where each student kept a record of number facts that she or he knew, and spent some time reinforcing these.

Building on a known fact (8 minutes)

The teacher and student worked on deriving number facts from one of the child's known facts. This provided the main teaching emphasis for the session.

Working with large numbers or problem solving (2 minutes)

The final minutes were spent either exploring what could be derived in terms of large numbers (for example, working on what double 400 must be if a student knew double 4) or putting the number facts being worked on into the context of a problem.

Discussion

Statistical analysis clearly demonstrates that the intervention strategies developed were successful. They substantially increased the quantity of number questions that the targeted students were able to answer correctly, and significantly improved the profile of the techniques used by the students to arrive at correct solutions.

A primary implication of these findings is that we do not need to wait for children to be 'ready' to be taught new strategies. Through carefully targeted teaching, students who have not developed these strategies for themselves can indeed learn them.

The analysis of the qualitative data raises questions about the extent to which low attainment is actually the result of some 'deficit' in the child. It seems, rather, to be something that is constructed between the teacher and student through neither of them

being totally clear about the expectations of the other. This is an important area for further research. As the sessions progressed some of the teachers commented more critically on the best use of teacher time in such circumstances, as well as on the implications for the classroom. As one teacher commented:

I used to just plan a topic, say multiplication, for half a term and hope that through a wide range of experiences, something would just 'stick'. I now realise that even in just ten minutes I can make a difference.

References

- Bennett, N., Desforges, C., Cockburn, A. & Wilkinson, B. (1984). *The quality of student learning experiences*. London: Lawrence Erlbaum.
- Bruner, J. (1986). *Actual minds, possible worlds*. London: Harvard University Press.
- Crooks, T.J. (1988). The impact of classroom evaluation practices on students. *Review of Educational Research*, 58(4), 438-481.
- Denvir, B. & Bibby, T. (2001). *Diagnostic interviews in number sense*. London: BEAM Education.
- Gray, E.M. (1991). An analysis of diverging approaches to simple arithmetic: preference and its consequences. *Educational Studies in Mathematics*, 22(6), 551-574.
- Hart, K., Johnson, D.C., Brown, M., Dickson, L. & Clarkson, R. (1989). *Children's mathematical frameworks 8-13: A study of classroom teaching*. Windsor: NFER-Nelson.
- Hughes, M. (1986). *Children and number: Difficulties in learning mathematics*. Oxford: Basil Blackwell.
- Steffe, L.P. & Cobb, P. (1983). Cognitive development and children's solutions to verbal arithmetic problems: A critique. *Journal for Research in Mathematics Education*, 14(1), 74-76.
- Walkerdine, V. (1988). *The mastery of reason: Cognitive development and the production of rationality*. London: Routledge.

Facilitating Children's Conceptions of Tens and Ones: The Classroom Teacher's Important Role

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In this chapter we address ways teachers can provide opportunities for their students to reason flexibly with collections of tens and ones. Recent research suggests that teachers are well advised to consider the role contextual problems might play in supporting such opportunities (e.g., Gravemeijer, 1994; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). Often, pictures of collections of tens and ones are juxtaposed with numerals as students are introduced to notions associated with place value. It is erroneously assumed that because students can speak in terms of collections of tens and ones, they can also mentally manipulate collections of tens and are ready to add and subtract two- or three-digit numbers. We and others (e.g., Cobb et al., 1997; Gravemeijer, 1994; Kamii & Housman, 2000; Labinowicz, 1985) suggest that children can reason with two-digit numbers using self-invented, sensible methods that often do not fit with standard notational methods. Further, at least with regard to two- and three-digit numbers, these less traditional methods are more desirable. Contextual problems can play a unique and important role in facilitating such methods.

In our discussion here, we continue to address these ever-pressing decisions of practice that classroom teachers face. Our view is that supporting children's flexible thinking about collections of tens and ones should precede or occur concurrently with providing instruction that focuses on addition of two-digit addition and subtraction. Further, like Cobb et al. (1997) and Gravemeijer (1994), we suggest that it is necessary to use informal situations or contexts that provide opportunities for students to develop more sophisticated ways of reasoning with quantities. Otherwise, students will often resort to counting methods, even when they may be able to move beyond those methods to reasoning with quantities.

As an aside, for a child to reason flexibly with collections of tens and ones, he or she must see a quantity as composed of collections of tens and ones interchangeably. The child can decompose collections of tens into individual countable objects and can recompose these countable objects into groups of tens and ones. For a quantity, say 45, the child can simultaneously see 45 as 4 tens and 5 ones, 3 tens and 15 ones, 2 tens and 25 ones, 1 ten and 35 ones, or 45 ones.

In the discussion that follows, we use excerpts from two lessons to illustrate the importance of the teacher's role as she develops and capitalises on informal situations that support students' arithmetical activity. As we share examples of students' explanations, we will discuss the ways the classroom teacher, Ms Jones, supported the students' reasoning about collections of tens and ones. In addition, we will note Ms Jones' role in making the students' thinking topics of discussions during the lessons. The excerpts that we use are taken from two of the many lessons that focused on the students' reasoning with pictorial collections. Initial discussions about manipulating tens and ones, such as the ones we offer here, contributed to the later discussions around addition and subtraction of collections up to 100. Before we share these examples, we first provide some background about the project classroom in which we conducted this second-grade teaching experiment.

Background

Prior to working with the classroom teacher, Ms Jones, we were aware that her practice did fit with an inquiry approach (Richards, 1991). She capitalised on children's thinking and often used their contributions to generate investigations in which the students explored open-ended problems. Students were expected to share their strategies and the thinking behind those strategies with the whole class. The other students were expected to listen and try to understand their classmates' thinking.

As one of the goals of the project, we developed a series of activities that we hoped would support the students' flexible manipulation of two-digit quantities in addition and subtraction situations. We adapted tasks from previous classroom teaching experiments conducted by the Purdue *Problem Centered Mathematics Curriculum Project* (Cobb, Yackel, Wheatley, Wood, McNeal, & Preston, 1992) and the *Mathematizing, Modeling and Communicating in Reform Classrooms Project* (Cobb, Yackel, & Gravemeijer, 1995). Using some of these previously developed materials, we collaborated with Ms Jones to design a context around candy that Ms Jones' Aunt Mary made and distributed at various community functions. (Ms Jones did have an Aunt Mary who made candy for family and friends.) As the students worked in this context, we planned for them to create physical collections that represented packages and pieces of candy. Later, they began to draw pictures to represent their actions with those collections. As a consequence of these early experiences, we conjectured that the students would use pictures to reason about addition and subtraction situations involving Aunt Mary's candy. Eventually, we thought that the students would use more formal ways of notating to represent their reasoning. As such, the informal situations would evolve into ways of notating and symbolizing that fit with more conventional ways of writing and interpreting addition and subtraction situations (c.f., Cobb et al., 1997; Gravemeijer, 1994).

The intent of early whole class discussions around this context was to establish the convention of packing candy in groups of tens and combining these groups with single pieces of candy (loose candy). Using this context, Ms Jones introduced several activities in which her students made packages and pieces of candy. The first of these activities focused on Aunt Mary's problem of organizing her candy-making so that she could easily keep track of how much candy she made, how many she distributed to people, and so on. Following a discussion about the possible ways Aunt Mary might pack the candy, the students, working in pairs, were given bags of loose multilink cubes to make packages of candy. They were asked to estimate how much candy was in their

bag and to package the candy so that if they were Aunt Mary they could efficiently determine the amount of candy she had. The students then shared their packaging ideas with the whole class. As a consequence of this activity, Ms Jones and the students established the convention that the packages would always contain ten pieces of candy.

These early discussions about how Aunt Mary might package the candy were very important. The students offered various suggestions about how Aunt Mary might package candy. They also discussed the need for her to consider other factors including what kind of containers and wrappers she should use. The students were so invested in helping Aunt Mary solve her problem that they suggested to Ms Jones that they write letters to share their ideas with Aunt Mary. The students' engagement in this informal situation and their understanding of Aunt Mary's problem of packaging candy was vital and made it possible for Ms Jones to introduce subsequent activities involving Aunt Mary's candy.

For a follow-up activity, Ms Jones used the overhead projector to show several of the students' journal entries. The students had drawn pictures in their journals to show how Aunt Mary's candy might be organised on her candy counter when she was interrupted by a phone call. Using students' pictures, Ms Jones asked the children to determine the amount of candy that was on the candy counter. As part of Ms Jones' pedagogical agenda, she planned to highlight the students' explanations as they combined pieces of candy to make packages. In so doing, she intended to make the newly established convention of creating packages of ten pieces of candy an explicit topic of discussion. With regard to the students' conceptualisations, she hoped they would make groups of ten that they could then act with in a variety of ways as they determined the amount of candy in the pictures. We now turn our attention to excerpts from the lessons.

Aunt Mary's Candy Counter

The two examples we share occurred during different whole-class discussions as the students determined how much candy was on Aunt Mary's candy counter. We use these discussions to illustrate the different ways the students could think about and evaluate (determine) the amount of candy in the pictures. We also highlight the ways in which Ms Jones facilitated the students' discussion within the scenario of Aunt Mary's candy.

For the first of these examples, Ms Jones showed the students' drawings of packages and pieces of candy on an overhead projector. These pictures were visible on a screen in the front of the room for all the students to see during the discussion. We enter the discussion after the students have talked about several pictures of packages and pieces of candy. As the discussion continued, Ms Jones showed Walter's picture of four packages and twenty-one pieces (see Figure 1). Referring to the number of loose pieces in Walter's picture, she explained:

Ms Jones: Okay, Walter's problem has my aunt packaging some things up. But, it looks to me as if she has a lot of pieces. She must have just emptied that pan out and started to wrap them up [before she was interrupted].

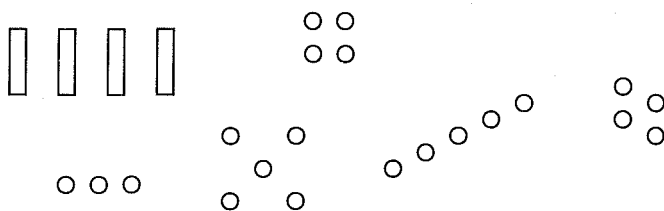


Figure 1: Walter's picture of sixty-one pieces of candy on Aunt Mary's counter

As Ms Jones set the stage for the ensuing discussion, notice that she spoke about how Aunt Mary's candy was arranged on the kitchen counter. That is, Aunt Mary has placed many loose pieces on the counter that she would need to package up. Her comment here is important because she provided opportunities for the students to imagine the situation and to understand how Aunt Mary will keep track of all the loose candy.

Following her comment, Ms Jones called on Yvette, who explained how much candy was on the counter. We enter the discussion after Yvette, who was standing at the screen at the front of the room, explained that she combined four packages to make forty. As Yvette explained her thinking, Ms Jones drew lines under the four packages and wrote the numeral forty under the packages. As the discussion continued, Yvette explained how she made another package from the loose pieces and added this package to the forty she already had:

Yvette: Then a ten right there. (Points to three groups of loose candy.) These three, that five, and these two.

Ms Jones: This five and these two (draws a circle around the three, five and two pieces of candy. See Figure 2). She has two up here, five [here], and that makes seven plus three more.

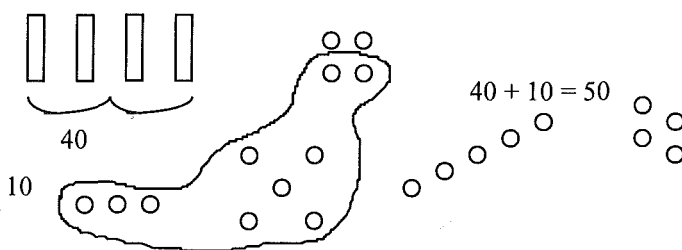


Figure 2: Ms Jones notated Yvette's explanation for adding 40 and 10 more

Yvette: Forty plus ten makes fifty.

Ms Jones: (Writes the numeral 10 beside the circled pieces and writes the number sentence $40 + 10 = 50$, see Figure 2.) So you have a ten up there and that makes fifty. Okay.

Yvette spoke of putting the single pieces of candy together to make another package. Ms Jones, for her part, circled and labelled the new package that Yvette made and wrote a number sentence to indicate that Yvette added this new package to the other four packages. That is, she highlighted the fact that Yvette made a collection of 10 pieces of candy to make another package. By doing so, she made it possible for Yvette and the other students to see this circled group as composed of ten pieces of candy that they might take as one package or a group of ten.

Following this exchange, Yvette explained that she added the remaining loose pieces to have a total of fifty-nine pieces. Several of the students disagreed with Yvette, which in turn prompted Yvette to reconsider her answer. She then changed her answer to sixty-one. Ms Jones, without indicating whether Yvette's first or second answer was correct, asked her to explain how she thought about the remaining candy:

Ms Jones: Okay, you looked at this (points to the group of five and four pieces of loose candy) and you had what?

Yvette: Umm, fifty-nine.

Ms Jones: Okay, you had fifty-nine (begins to circle around the five and four pieces. See Figure 3). And then what did you realize?

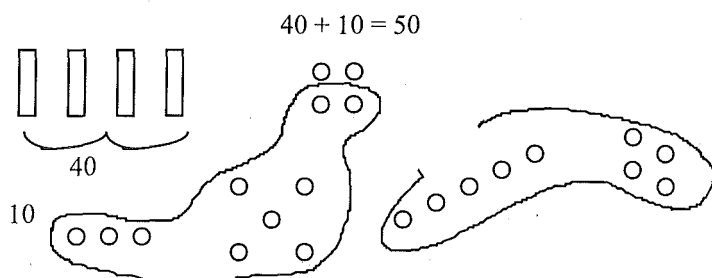


Figure 3: Ms Jones began making another package by circling nine pieces of candy

Yvette then pointed to the two remaining pieces and explained that she needed to add them to the nine remaining pieces. The discussion continued as Ms Jones asked Yvette what she planned to do with these loose pieces of candy:

Ms Jones: Okay, so what are you going to do? How are you going to package? Can you package up another package? Can you make another package?

Yvette: I think this, this one should be alone (points to one of the two remaining pieces). One needs to be alone.

Ms Jones: Okay, so I take this one in here. Do I have another package (continues to draw line around one of the two remaining pieces to make a package and writes 10 outside of 10 circled pieces. (See Figure 4)?

Yvette: (Nods yes.)

Ms Jones: So I have a ten there right?

Yvette: Yeah.

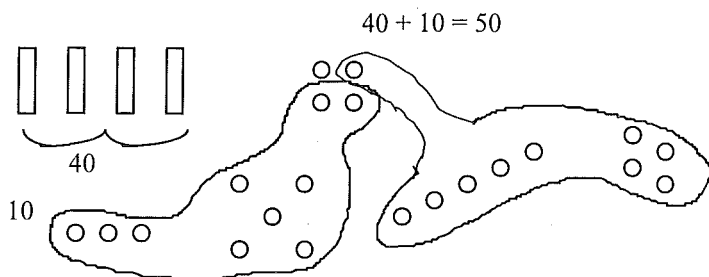




Figure 4: Ms Jones notated Yvette's making of another package

By asking if Yvette could make another package, Ms Jones continued to encourage Yvette, and perhaps some of the other students, to make packages whenever possible. By doing so, Ms Jones made it possible for Yvette to explain that she combined one of the leftover pieces with the nine loose pieces to make another package. After Yvette made the sixth package, Ms Jones again made her thinking explicit as she circled ten pieces to make another package.

This classroom example illustrates how the students began to think about packages and loose pieces of candy. These early discussions about making packages with pieces of candy proved to be important. For some of the students, making these packages, first with the multilink cubes and later with pictures, was useful in helping them to conceptualise collections of ten that they eventually could treat as groups of ten. Yvette's explanation, for instance, may have provided an opportunity for other students to consider the pictorial candy that Yvette could act with to make and evaluate collections of ten. If need be, students could continue to use counting strategies to make these packages. However, as we have shown here, students who did not need to count by ones could increment by tens and ones.

Our second example occurs during a subsequent lesson several days later. We use this example to further illustrate the various ways the students could reason with the pictures of candy. In addition, we continue to point out the teacher's role in facilitating the students' thinking and acting with the pictures. The second example is drawn from a discussion that occurred as the students shared the various ways that they combined the candy. The task was posed as follows:

Aunt Mary has   on the counter. She makes 19 more pieces.
How many candies does she have in all?

Ms Jones called on Janet who came to the overhead projector and drew how much candy would be on Aunt Mary's counter (see Figure 5). Interestingly, once Janet drew her picture, discussion shifted to evaluating the packages and pieces (similar to the Yvette example shared earlier). We enter the discussion as Ms Jones asked Janet to explain her thinking:

Janet: I'd count ten, twenty (pointing to packages), twenty-one, twenty-two, twenty-three, ... forty-two (counts on by ones as she points to each of the 19 loose pieces).

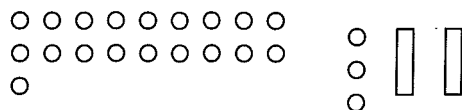


Figure 5: Janet's picture of candy that is on Aunt Mary's counter

Here, Janet counted on from 20 by ones to determine the amount of candy in her drawing. Because she had drawn individual loose pieces of candy to show the 19 she must add, we might have predicted that she would count the individual circles. Also, we presume that she may have used a number fact to organise the 19 pieces. Note that she arranged the 19 circles in two rows of 9 and a singleton. Interestingly, at least as she counted, she did not refer to 19 as one package of 10 and 9 loose candy, nor did she increment by 10 and then add on 9 more. At least at this point in the discussion, the convention of packaging candy in groups of ten when possible did not appear particularly significant for her. As the discussion continued, we see how Ms Jones capitalised on Janet's contribution:

Ms Jones: Okay, she got forty-two for her answer. I have a question for you, Janet. Knowing that Aunt Mary always makes packages after she has all the candy out on the counter, is there a way you could have done this faster?

Janet: Yeah.

Ms Jones: What could you have done?

Janet: I could have done this (makes a circling motion around the loose pieces).

Ms Jones: Okay. Could you do that for me? (Looks at the rest of the class and says) One of the things Aunt Mary always does, she said, is she goes ahead and makes the packages as soon as she can, because that makes more sense to her. So I'm asking Janet to, if there's a way that she could do that too.

Janet: (Circles two groups of 10. See Figure 6.)

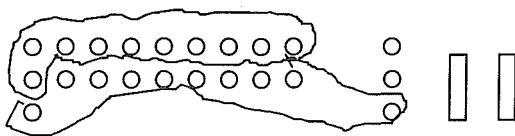


Figure 6: Janet circled two groups of 10

Note how Ms Jones immediately brought the discussion back to the context of Aunt Mary's candy and reminded Janet and the other students that Aunt Mary packages candy whenever possible. More importantly, she asked Janet if she could offer a different, more efficient way to determine the amount of candy. In response to Ms Jones' question, Janet began circling groups of ten pieces of candy.

The discussion continued as Ms Jones and Janet talked about the groups that Janet had made:

Ms Jones: Okay. If you grouped these here, how would they look as packages (points to circled pieces)?

Janet: (Draws two packages beside existing packages. See Figure 7.)

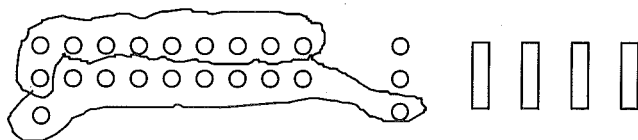


Figure 7: Janet drew two packages to show two new packages

Ms Jones: And how many extras will she have there?

Janet: (Pointing to two uncircled pieces) Two.

Ms Jones: She still has the two there. So if we wanted to rewrite it, we could rewrite it this way. (Draws a line to the left of the 4 packages and draws two pieces. See Figure 8.) Do you agree this is what she would have?

Students: Yeah.

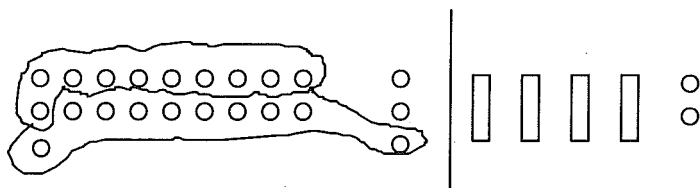


Figure 8: Ms Jones drew two loose pieces and separated the circled pieces from the final picture of four packages and two pieces

Ms Jones' question to Janet at this point in the discussion was particularly important. By asking Janet to consider how the circled pieces might look as packages if they were grouped, she prompted Janet to draw two new packages. In doing so, Janet had the opportunity to make a pictorial representation of the groups of candy she circled.

This second example illustrates again the teacher's important role in capitalizing on situations that might support student learning. This exchange may have provided Janet and some of the other students an opportunity to reconsider the collections in terms of tens and ones. Whereas the convention for packaging candy in groups of tens whenever possible had been discussed for several days, some of the students did not evaluate the pictures by making groups of tens and then combining packages and loose

candy. For instance, in this example, Janet counted on to solve the task. Although her explanation was not incorrect, Ms Jones decided to capitalise on this instance to provide Janet the opportunity to reconsider her pictures in terms of making packages of ten.

When we recall how Yvette evaluated Walter's picture in our previous example, it becomes clear that there were a variety of ways the students could participate in and contribute to these discussions. Yvette could routinely evaluate collections of tens and ones when she made packages. By way of contrast, Janet experienced some difficulty in evaluating collections of tens. Yvette's and Janet's explanations during interviews conducted several days before these discussions are also consistent with our observations during whole class discussions. Although Yvette could both make and mentally combine tens as she referred to the pictures of candy on Aunt Mary's counter, Janet appeared to have some difficulty evaluating the pictures unless she counted by ones. Even in the above example, neither Ms Jones nor Janet spoke of the four packages as forty and two more to make forty-two. Whether or not the students needed to count individual pieces of candy, these activities provided an opportunity for all of the students to solve the tasks in ways that fit their current ways of knowing. It is for this reason that the scenario about Aunt Mary's candy provided them opportunities to act sensibly with collections of tens and ones in a variety of ways. As such, discussions such as the ones we have highlighted here were essential and contributed in part to the progress students eventually made. As an aside, we note that later in the school year, Janet could routinely evaluate pictorial collections by combining tens and ones.

Final Comments

Although we stress the importance of such activities, we would be remiss if we suggested that these types of activities are the only tasks needed for students to reconceptualise collections of ten objects as one unit of ten. A great deal of care is necessary in designing the follow-up activities that might provide students with opportunities to move beyond acting on pictorial representations of Aunt Mary's candy. A range of activities is necessary to build additional experiences to support the students' continued learning. For instance, later in the year, Ms Jones used the overhead projector to flash pictures of candy for two or three seconds (c.f., Cobb et al., 1992). For these subsequent tasks, the students were asked to determine how many pieces of candy they saw and how they saw the candy. Suppose Ms Jones posed the task pictured in Figure 9.

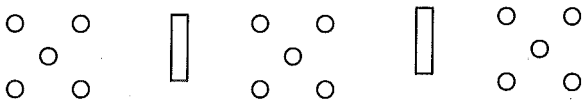


Figure 9: A 'flashing' task

To explain how they figured out how many were visible, the students needed to mentally combine pieces of candy to make packages when possible. A student might explain, for instance, that she combined two groups of five to make a package (the left and middle groups), added this package to the two packages she already had to make thirty, and then added the remaining five pieces to make thirty-five. Although this

student's explanation seems similar to the explanations that were presented previously, this task is much more taxing for the student. Unlike the previous examples, the student must be able to put the candy together mentally to make packages once the packages are no longer visible. As such, this activity requires the student to mentally manipulate the groups of candy to make collections of ten, then combine these new collections with the other collections of ten, and finally combine these packages with the remaining loose pieces of candy. In order to determine the amount of candy, the student must keep these images in her mind. Thus, much more conceptual work is required.

These and other variations of the flashing tasks provided the students opportunities to manipulate collections and later two-digit quantities mentally. Subsequently, these tasks along with others contributed in part to the success the students had in solving two- (and three-digit) addition and subtraction tasks. The students' experiences with breaking packages up and putting pieces together to make packages were critical to the success they had in mentally adding and subtracting two-digit numbers. As such, the students' flexible manipulation of quantities had its origins in these initial experiences with Aunt Mary.

As Ms Jones presented these and other activities, she continued to highlight the students' thinking. She also continued to use more conventional notational methods as she re-described aspects of their thinking. In doing so, she and her students established ways to conceptualise and communicate mathematical ideas when solving addition and subtraction problem situations.

By introducing the context of Aunt Mary's candy, Ms Jones provided opportunities for her students to make sense of their own and others' thinking. Informal contexts of these types in which students can engage deeply over time offer much stronger support for students' thinking about collections of tens and ones than traditional methods. When the context is ongoing, time can be taken initially to help the children become fully engaged in that context, and they are freed from the need to 'buy into' a new context each day or whenever they are introduced to a new problem. By engaging the students in the context, the teacher enables them to use their real-world knowledge to construct abstract mathematical ideas.

References

- Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Symbolizing and mathematizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner & J.A. Whitson, (Eds.), *Situated cognition theory: Social, semiotic, and neurological perspectives* (pp. 151-233). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Cobb, P., Yackel, E., & Gravemeijer, K.P.E. (1995). *Mathematizing, modeling, and communicating in reform classroom. Progress report*. Unpublished manuscript.
- Cobb, P., Yackel, E., Wheatley, G., Wood, T., McNeal, B., & Preston, M. (1992). *Purdue problem centered mathematics curriculum project*. Unpublished manuscript.
- Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education*. Utrecht, The Netherlands: CD-β Press.

- Kamii, C.K., & Housman, L.B. (2000). *Young children reinvent arithmetic: Implications of Piaget's theory* (2nd ed.). New York: Teachers College Press.
- Labinowicz, E. (1985). *Learning from children: New beginnings for teaching numerical thinking*. Menlo Park, CA: Addison-Wesley.
- Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld, (Ed.), *Radical constructivism in mathematics education* (pp. 13-51). The Netherlands: Kluwer Academic Publishers.

Part 2

The Role of Calculators

Introduction

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Part 2 provides three insights into the use of calculators in schools, drawn from experience of projects in Australia and the United Kingdom. The potential value of calculators for developing numeracy and number sense is still too little understood.

Sparrow and Swan take the view that a calculator aware curriculum must plan for the development of expertise in calculator use. They declare that schools should not just passively allow calculators, but rather they should have some specific educational tasks to perform. They propose a list of basic skills related to using a calculator and suggest a sequence of development of these skills in the primary school. They emphasise the use of the calculator as a computational tool and as a learning aid, and contend that, far from hindering children's thinking, calculators can be used to enhance it; but they indicate the need for teachers to justify their use if they are to be used efficiently.

Groves reports on a major calculator project which she conducted in Victoria, involving over 1000 children in Grades K-4 over four years. She describes the differing roles played by the calculator and provides examples of children's activities and progress in key areas. She concludes that both children's competence and understanding of numbers and computation were increased by prolonged calculator use, and reflects on consequences both for our increased awareness of children's potential and for the curriculum of young children.

Ruthven draws lessons from the *Calculator-Aware Number* (CAN) project directed by Hilary Shuard in the United Kingdom. He suggests that the key issue in the project was a shift of emphasis from written computation to mental computation, with calculators acting as a catalyst. He gives a balanced view of the strengths and weaknesses of the project, which provide valuable lessons for today. Embedded in both the Victorian and the British calculator projects described here is the role of teachers: they do not simply execute the strategies and activities devised by project leaders; rather they have the responsibility of devising and evaluating activities which they consider will foster the intentions of the project. This is an admirable philosophical standpoint but it has some clear weaknesses in its expectations of teachers. Ruthven's analysis of the impact of the British National Curriculum, with its restrictive requirements on a very open project, is illuminating.

Techno-ignorant, Techno-dependent or Techno-literate? A Case for Sensible Calculator Use

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Edith Cowan University, Perth

Put down that calculator, stupid! (Gelernter, *New York Post*, May 21, 1998)

Calculator's days may be numbered. (Bentham, *International Express*, August, 1994)

The result of knee-jerk reactions like those above, aimed no doubt at ensuring that children do not become techno-dependent, could be a decade of children who are almost techno-ignorant. These and other similar headlines highlight an interesting phenomenon in mathematics education. After a supposed boom in the use of calculators in the classroom, serious questions are being asked about their value. It is claimed that the backlash has, in some cases, led to schools abandoning their use. This wholesale banning of calculator use in primary school classrooms in deference to mastery of pencil and paper standard algorithmic procedures and instant recall of number facts may appeal to many people, but not to all.

We will argue that what is needed is neither a generation of techno-dependent nor one of techno-ignorant citizens, but one that is techno-literate and able to use the power of sophisticated machines in sensible ways when it is appropriate to do so. We will consider what it is to be techno-dependent and techno-ignorant. Suggestions will then be offered for ways to develop not only techno-literate but also mathematically more able citizens. To be techno-literate in this case is to be able to use a calculator efficiently, effectively and appropriately in mathematical situations. The article will consider the implications of this aim on teaching and learning mathematics in the classroom and analyse the skills of calculator use, which need to be taught, and how the calculator might be used to enhance the development of number sense.

Back-to-Basics: The Answer to Techno-dependence?

Stories of techno-dependent individuals abound. Almost everyone has an anecdotal account of the shop assistant who cannot add even the simplest of numbers without resorting to the cash register or children who use the calculator for every calculation. Certainly these are examples of techno-dependent people. Many assume the reason for this lack of arithmetic ability and techno-dependence is the use of calculators in classrooms. For them, the way to stop techno-dependence is to ban the use of

calculators in classrooms and give children a larger serve of basic arithmetic. But there is a problem with the popular conception that a style of teaching which emphasises procedures, speed and quick-fire mental calculation recall will redress the problem. This philosophy may actually inhibit the development of the most powerful mental strategies (Ruthven, 1998).

Adults and some children become techno-dependent, not because of calculator use in primary schools—for many of them calculators were not present or even available in their primary school years—but because they were not allowed to develop strategies for mental calculations and to develop number sense. Calculator use is not the villain. The influence of narrow teaching in many classrooms, which emphasises instant recall and standard, rote learned methods of calculation, is much more likely to create this state of techno-dependence. Shop assistants reach for the calculator because they have no mental computation strategies available to them to calculate the answer to the problem. Ralston (1999) has suggested that there is evidence that many children have remained essentially innumerate under the classic pencil and paper curriculum.

The way forward is not therefore a return to a rigid pencil and paper regime, nor is it open slather with calculators but rather the need to develop in children the ability to use competently and confidently the tools of modern society—sensible calculator use and mental methods.

Techno-literate: The Way Forward?

We restrict our definition of using technology to a typical basic four-function calculator. So, what is a techno-literate child? As a starting definition we offer the following:

A techno-literate child is able to make an informed choice about the appropriateness of using, or not using, a calculator for a given computation. The child is able to use a calculator effectively and efficiently for everyday numeracy needs. The child is able to make judgements about the answer displayed.

One aim of an entitlement curriculum should be to develop children's facility in the use of the tools which society uses. For example, almost all calculation in employment is carried out by calculating machines (Fitzgerald, 1986). One could surmise that this is even more pervasive a decade or two later with an associated need for techno-literate and techno-competent workers. There are few things we can predict about life and the needs of society in future times. We can be sure however, that it will be different from our present experience and that people will need to be flexible, rather than narrow and constrained in their thinking. Back to the basics in the twenty-first century requires a broader view of mathematics than one of proficiency with standard procedures, one that refuses to divorce computational practice in its widest sense from thinking and reasoning and one that keeps problem solving squarely at the forefront (Burns, 1998). If we are to consider the future of our children rather than re-live our past then we need to move forward; to go, not back-to-basics, but beyond-the-basics (Lappan, 1998).

This new generation of adults will be empowered to make choices related to context, calculation strategy, efficiency and method, rather than become a techno-dependent generation which uses a calculator for almost all jobs because they have not been given the necessary calculation strategies to do otherwise.

Recent debates on the issue of numeracy have highlighted the connection between numeracy and technology.

There is no doubt that the increasingly technological society in which we live is making different and greater demands on our numeracy. This is true in our lives at work or in education, at home and as citizens. The workforce which this country needs for the next millennium is one which is technologically capable. (AAMT, 1997)

One way to develop a power with numbers is through a sensible use of calculators by children involving thoughtful, planned and integrated activities with their teachers. In the next section we consider the skills of calculator use needed to help children become techno-literate, as part of the debate about how to integrate calculators in the classroom for worthwhile purposes.

What Calculator-related Abilities are Needed by Children to Become Techno-literate?

For many children and adults most of the keys on even the simplest of calculators, for example the memory functions, are not used because the person has had no training in how to use them. A diet of mathematics which does not acknowledge the presence and power of calculators and one which does not teach children how to use the calculator effectively and efficiently may, in fact, continue to produce members of society who are almost techno-ignorant and mathematically impoverished. Possibly part of the problem of limited calculator use in primary schools is because teachers are unaware of the skills and concepts which could be developed by children. Certainly, Sparrow and Swan (1997) and the Schools Curriculum and Assessment Authority (1997) noted this aspect of classroom life.

There are two aspects to the development of informed calculator use, namely, the operation of the machine and the interpretation of the display (SCAA, 1997). An important feature of being techno-literate is to be able to use the features of the calculator appropriately. Simply putting calculators into children's hands without planned development, and then expecting them to acquire the knowledge of the functions will not work (Ralston, Reys, & Reys, 1996). This point is also emphasised by Her Majesty's Inspectors of schools in the UK:

There are skills in using a calculator, which need to be taught and learnt. A policy of allowing pupils to use a calculator is not enough (DES, 1988).

The connection between number sense and using calculators in efficient ways is not new. Girling (1977), and Bell, Burkhardt, McIntosh and Moore (1978) offered the following:

- checking the answer for appropriateness in case a mistake has been made in the keying section;
- the need to understand the relative size of numbers;
- the ability to perform mental calculations at speed, that is at the level of at least single digit arithmetic;
- a good understanding of place value and decimals; and
- the ability to estimate.

The suggestions here are concerned mainly with understanding the result of a calculation on a calculator. Thus, not only is the knowledge of key functions important, but also how to interpret the outcome of a calculation on the display. The list in Figure 1 isolates the skills related to operating a basic calculator effectively. The features have been drawn from recent recommendations and research findings.

Appropriate mental checking strategies via estimation and approximation.
How to use a calculator sensibly for various non-school purposes.
How to interpret the display.
How to use the keys efficiently and effectively (for example the memory, square root and percentage keys).
How to check the size and appropriateness of answers and offer reasons why the answer is useful.
Know and apply the order of operations.
Make decisions between mental, written and calculator approaches to a specific calculation.
Interpret the display with rounding.
Use with fractions; e.g. converting fractions to decimal form for calculation.
How to read the display and enter numbers on the calculator from other sources.
<div>How to use the constant function to:</div> <ul style="list-style-type: none"> • count forward • count backward • work with multiples and divisors • perform repeated addition and subtraction

Figure 1: Skills related to operating a basic calculator

While the skills have been detailed, there is no suggestion that each skill should be explicitly taught in isolation via a series of separate calculator lessons. It is envisaged that there would be systematic and planned teaching of the various functions of the calculator embedded in activities related to the development of number sense rather than haphazard discovery. Here the focus is on the mathematical principles inherent in the task rather than a specific calculator skill (van den Brink, 1993). Children need to be helped to see the relationship between the mathematical concept and its operationalisation on the display screen (Ruthven & Chaplin, 1997). Many of the skills offered would be suited to ideas being developed with younger children, whereas others are more appropriate for older children, especially if a fraction calculator or the newer, less complex graphing calculators were employed with the upper primary children (Kissane, 1997). Children also need to understand the limitations of the calculator and devise strategies to work around these limitations (Swan, 1998). Many of these

strategies, however, need reinforcing throughout the primary school years and are not just restricted to year levels. Thus the development of calculator skills is not seen in a strictly linear way.

An argument for indiscriminate or laissez-faire implementation of calculators is not being offered. In fact, it has been argued (Duffin, 1994) that a calculator used thoughtlessly and for all calculations can have the effect feared by those who distrust them. If the skills of computation are not practised they may soon be lost. A school policy on sensible calculator use and the integration into mainstream mathematics would highlight and note the dangers.

Some classroom practices are seen as dangerous, sending the wrong message to children or just plainly a waste of time. For example, Higgins (1990) points out:

[I]n some situations we should require that unimaginative mathematics teachers not use calculators. Simply keying a problem from a mathematics textbook into a calculator and pressing the equal key may very well be a way to avoid thinking. Teachers who cannot imagine other ways to use calculators in classrooms should be required to stop using calculators in mathematics classrooms at once. (p. 4)

The common practice of using the calculator as an electronic answer book to check work done by pencil and paper methods is, according to Reys and Reys (1987), counterproductive. They cite several reasons for this including:

- providing an answer is more efficient,
- it does not reflect real world practice, and
- it implies that calculator use is cheating or not appropriate.

Rousham (1995) warns against the novelty effect of the calculator. This often occurs when calculators are given to a class, usually of older children, who have had little or no contact with calculators in the school context. Unless there is considerable thought given to the role of the calculator, children will tend to use them for everything—even for the simplest calculation. Children need time and access to calculators as well as thoughtful activities to help them cross the ‘familiarity threshold’ (Rousham, 1995).

The development, in an integrated way, of skills for calculator use has been identified and aspects of classroom methods questioned. But what principles and guidelines for classroom use are appropriate? The next section sets out to consider this question.

The Calculator as a Vehicle for Learning Mathematics

Another aspect that appears to be missing from many primary classrooms is the use of the calculator as a teaching and learning aid. The major issue is how best to use calculators in combination with other strategies to improve the teaching and learning of mathematics. Here the role of the calculator is much the same as that of Multibase Arithmetic Blocks, Unifix cubes, squared paper and other teaching aids—to help children understand what is happening and to make connections between mathematical ideas. This notion of children making connections between mathematical ideas is a strong one. The most effective teachers of numeracy are those that help children connect pieces of mathematical knowledge and build a network of interconnections (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997). We argue that the calculator can

be used in this way to help children make powerful connections between ideas and develop the necessary flexibility for number sense. Possible reasons for the absence of approaches such as these were proffered earlier; namely the lack of a coherent and explicit policy linking calculator use and number sense development, the stranglehold of textbooks and other commercially produced materials, and teachers' personal lack of knowledge of their potential.

Principles for how teachers may help children develop these connections are offered in Figure 2. Here guidelines derived from recommendations in the literature on mathematics education are outlined.

Identify the purpose for the use of the calculator in the particular activity.
Have a school policy for the use of calculators and for the development and consolidation but not replacement of number concepts and skills.
Help children pay attention to efficient and appropriate computational strategies, including calculator, mental and written methods and to decide when or if to use a calculator (or other method) when faced with a calculation.
Use calculators as a time-saving device to allow children to explore and demonstrate concepts.
Use calculators to explore number patterns and place-value relationships, to enable children to develop a feel for the embedded structure of the number system.
Use calculators to allow children to work with realistic data rather than contrived or unrealistic figures.
Use them with ideas of estimation and approximation and to develop and consolidate children's mental and written strategies for number work.
Use them to require and allow children to explain and justify their ideas.
Begin to develop children's number sense through the use of calculators before standard algorithms are taught.
Help children 'smash-up' and 'break-down' numbers with decomposition, distribution and compensation to develop and refine mental strategies.
Have children develop the habit of estimating first and using the estimate-calculate-check procedure and explain why their answer on the calculator is reasonable.
Do not restrict the children to working only with numbers within the text book or syllabus range.
Do not teach calculator use in isolation but integrate its features and functions into other contexts.
Incorporate calculator use into assessment.

Figure 2: Guidelines for teachers

A fundamental question that could be asked by teachers before they embark on an activity with children, assuming there is a good mathematical or educational reason for undertaking it, is, "What is the role or purpose of the calculator in this activity?" The teacher could adopt a rubric often used with computer tasks. Here the teacher analyses the task in relation to the following three questions:

- Can the calculator do aspects of this task quicker?
- Can the calculator do aspects of this task in better ways?
- Can the calculator do aspects of this task that could not be done without it?

If one or more answers to the questions is 'yes', then it may be worthwhile using the calculator.

Calculator Use and Developing Number Sense

In their framework for number sense McIntosh, Reys and Reys (1992) outlined three major components, namely:

- knowledge and facility with numbers,
- knowledge and facility with operations, and
- applying the above points to computational settings.

It is this sense-making of numbers and knowledge of the process of arithmetic that is crucial to further learning and success in mathematics. Thoughtful and planned calculator usage can play a key role in this sense-making by children. The use of calculator activities integrated into a number sense approach to computation may be a way to develop numerate as well as techno-literate children.

Pupils whose experience had been strongly shaped by the calculator aware approach to number were more liable to use mental calculation in tackling problems presented to them and less prone to fall back on the use of written or calculator methods. They made greater use of mental strategies based on decomposition, distribution and compensation. Heavy dependence on written and calculator methods was extremely rare among such pupils. (Ruthven, 1998, p. 22)

A suggested developmental sequence of basic calculator skills is presented in Figure 3. Developing number sense ideas will help children to use the calculator more effectively and at the same time to become less reliant on it and trusting of it. Bobis (1991) noted that good estimators usually placed greater trust in the calculator's answer than their own estimate. Thus, she argued there is an even greater need to deliberately nurture estimation skills to overcome this tendency—an aspect of techno-ignorance. Many of the estimation methods suggested by Swan (1996), for example using an alternative operation, estimating the order of magnitude, or using knowledge of patterns, are closely connected to aspects of number sense.

Age Group	Calculator Skills
Lower primary K-2	<ul style="list-style-type: none"> • key recognition 0-9 • key recognition +, -, x, =, divide • key recognition 'clear' • matching written and display styles for numbers • use of constant key • read display • change an incorrect entry • can use calculator related language
Middle primary 3-4	<ul style="list-style-type: none"> • using the constant function for repeating the same calculation many times • the estimate-calculate-check rubric • interpret the decimal part of an answer • aware of rounding and truncating aspects of calculators
Upper primary 5-7	<ul style="list-style-type: none"> • use of % key • use of memory • use of toggle key • possible development from a fraction calculator • use of negative numbers • use of square root key • use of iterative strategies • use of the correct key sequence for calculations with more than one operation • select whether to use mental, written or calculator approach

Figure 3: Calculator skills: A possible development in the primary school

Conclusion

The messages that are coming from research and official documents related to calculator use and learning mathematics in the primary school are saying that calculators must be used to:

- learn how to use a calculator effectively and efficiently;
- help children make appropriate decisions between calculation methods at the point of use;
- help children build, connect and test mathematical ideas, and
- support rather than replace mathematical thinking.

Gray and Pitta (1997) have noted that:

Calculators can give children an insight into numerical patterns and relationships that are hard to discern if children are constrained by the use of lengthy counting procedures or the knowledge of isolated number combinations. (p. 39)

A deep intuitive understanding is built up through continual exposure to explorations in number (Groves & Stacey, 1998). But as well as this use of calculators to develop connectedness and mathematical ideas, children need to learn how to use a calculator sensibly and accurately. They need to learn how to use one to import data, interpret results and, especially at the older primary school ages, perform the complex operations of which calculators are capable (Ainley, 1996).

One of the fundamental obstacles to the integration of calculator use in primary schools is the mindset of many teachers with regard to mathematics and mathematics teaching. When there is a move from the orthodoxy of seeing mathematics as purely the replication of standard methods and the instant recall of number facts based on the textbook, to a vision of children exploring rich mathematical tasks, then the calculator will be incorporated as a learning vehicle for children rather than be restricted to finding answers to computational exercises. Evidence from the *Calculator-Aware Number* project (Shuard, Walsh, Goodwin, & Worcester, 1991) and the *Calculators in Primary Mathematics* project (Groves, 1994) indicate that the sensible and informed use of calculators can act as a catalyst for fundamental change in mathematics teaching in the primary classroom. This is supported by Ruthven (1995) who states:

A more considered use of calculators is probably the most realistic medium term strategy for bringing distinctive opportunities for sustained use of computational technology to teachers and pupils across the educational system. (p. 250)

As a result of being techno-literate, children will have the power to make informed and techno-appropriate choices about calculations.

References

- Ainley, J. (1996). *Enriching primary mathematics with IT*. London: Hodder & Stoughton.
- Askew, M., Brown, M., Rhodes, V., Wiliam, D. & Johnson, D. (1997). *Effective teachers of numeracy: Final Report*. London: King's College.
- Australian Association of Mathematics Teachers. (1997). *Numeracy = Everyone's business*. Adelaide: AAMT.
- Bell, A. W., Burkhardt, H., McIntosh, A., & Moore, G. (1978). *A calculator experiment in a primary school*. Nottingham: Shell Centre for Mathematical Education.
- Bentham, M. (1994). Calculator's days may be numbered. *International Express*, August, 1994.
- Bobis, J. (1991). Using a calculator to develop number sense. *Arithmetic Teacher*, 38(5), 42-45.
- Burns, M. (1998). Math for the 21st century—Back to basics? *Math Solutions*, 23, Summer.
- Department of Education and Science (1988). *Mathematics for ages 5-16*. London: Department of Education and Science.

- Duffin, J. (1994). Calculators: Do they inhibit or enhance mathematical thinking? *Micromath*, 10(3), 25-28.
- Fitzgerald, A. (1986). *New technology and mathematics in employment*. Birmingham: University of Birmingham.
- Gelernter, D. (1998). Put down that calculator, stupid! *New York Post*. May 21, 1998.
- Girling, M. (1977). Towards a definition of basic numeracy. *Mathematics Teaching*, 81(4-5), 13-14.
- Gray, E. & Pitta, D. (1997). Changing Emily's images. *Mathematics Teaching*, 161, 38-41.
- Groves, S. (1994). The effect of calculator use on Third and Fourth Graders' computation and choice of calculating device. In J. da Ponte and J.F. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education* (pp. 33-40). Lisbon: PME.
- Groves, S. & Stacey, K. (1998). Calculators in primary mathematics: exploring number before teaching algorithms. In L. Morrow and M. Kenny (Eds.), *The teaching and learning of algorithms in school mathematics* (pp. 120-129). Reston, VA: NCTM.
- Higgins, J.L. (1990). Calculators and common sense. *Arithmetic Teacher*, 37(7), 4-5.
- Kissane, B. (1997). Growing up with a calculator. *Australian Primary Mathematics Classroom*, 2(4), 10-14.
- Lappan, G. (1998). NCTM responds to "Math Wars" report. *NCTM News Bulletin*, 35(1), 1-4.
- McIntosh, A., Reys, B. & Reys, R. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics*, 12(3), 2-8.
- Ralston, A. (1999). Let's abolish pencil and paper arithmetic. *Journal of Computers in Mathematics and Science Teaching*, 18(2), 173-194.
- Ralston, A., Reys, R. & Reys, B. (1996). Calculators and the changing role of computation in elementary school mathematics. *Hiroshima Journal of Mathematics Education*, 4, 63-71.
- Reys, B. & Reys, R. (1987). Calculators in the classroom: How can we make it happen? *Arithmetic Teacher*, 34(6), 12-14.
- Rousham, L. (1995). Can calculators make a difference? In J. Anghileri (Ed.) *Children's mathematical thinking in the primary school* (pp. 92-109). London: Cassell.
- Ruthven, K. (1995). Pressing on towards considered calculator use. In L. Burton & B. Jaworski (Eds.), *Technology in mathematics teaching* (pp. 231-252). Sweden: Chattwell-Bratt.
- Ruthven, K. (1998). The uses of mental, written and calculator strategies of numerical computation by upper primary pupils within a calculator aware number curriculum. *British Educational Research Journal*, 24(1), 21-42.
- Ruthven, K. & Chaplin, D. (1997). The calculator as a cognitive tool: upper primary pupils tackling a realistic number problem. *International Journal of Computers for Mathematical Learning*, 2(2), 93-124.
- Schools Curriculum and Assessment Authority (1997). *Discussion paper No.9: The use of calculators at key Stages 1-3*. London: SCAA.

- Shuard, H., Walsh, A., Goodwin, J. & Worcester, V. (1991). *Calculators, children and mathematics*. London: Simon & Schuster.
- Sparrow, L. & Swan, P. (1997). *Calculator use in Western Australian primary schools*. Perth: MASTEC, Edith Cowan University.
- Sparrow, L. & Swan, P. (1999). Do children really have a choice when calculating? In J.M. Truran and K.M. Truran (Eds.), *Making the Difference*. Proceedings of the 22nd Annual Conference of the Mathematics Education Research Group of Australasia (MERGA). Adelaide: MERGA.
- Swan, P. & Sparrow, L. (1997). Impediments to the use of calculators in primary schools. In N. Scott and H. Hollingsworth (Eds.), *Mathematics: Creating the future* (pp. 283-288). Melbourne: AAMT.
- van den Brink, J (1993). Different aspects in designing mathematics education: three examples from the Freudenthal Institute. *Educational Studies in Mathematics*, 24(1), 35-64.

Calculators, Computation and Number Sense: Some Examples from the Calculators in Primary Mathematics Project

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It is important to understand that there is no significant research that suggests that calculator use at any level is harmful to mathematical development or that pencil-and-paper arithmetic, a skill with rapidly declining practical value, is necessary or even particularly useful for later mathematical development (Ralston, 1999, p. 2).

Calculators should be banned from American elementary schools ... the calculator subtly undermines the whole math curriculum ... "once the calculator goes on", says Mike McKeown, a geneticist at the Salk Institute in San Diego, "the brain goes off, no matter what we hope" (in Gelernter, 1998).

Rather than subsiding, Dick's (1988) *continuing calculator controversy* continues to rage. While authors such as Duffin (2000) argue that calculator use should be integrated into the earliest school years, Klein (2000) claims that early calculator use undermines conceptual understanding. Moreover, in the United Kingdom, despite the evidence in favour of calculator use contained in the Schools Curriculum and Assessment Authority's (1997) own discussion paper, calculator use in the early years of primary school has all but been banned.

David Gelernter, a professor of computer science at Yale University, in his first Commentary for the *New York Post* (Gelernter, 1998) clearly articulates a somewhat extreme version of the anti-calculator position. The position is that, first, school mathematics is, and should be, about the teaching of procedural skills: "Most people have no use for 'mathematical concepts' anyway—arithmetic yes, group theory no." Second, written computation is seen to be at the heart of arithmetic: "If you haven't mastered basic arithmetic by hand, you can't do arithmetic at all—with or without calculators ... if you can't do arithmetic manually you can't do it mentally; and you will need to do rough mental arithmetic all the time." And finally, the only role for calculators in schools is assumed to be as a crutch to enable children to avoid the need for learning: "To be educated is to master a body of facts and skills and have them on-call 24 hours a day When you hand children an automatic, know-it-all crib sheet, you undermine learning—obviously. So let's get rid of the damned things".

Not only does the anti-calculator position ignore the ample research evidence showing that calculator use does not lead to a deterioration of children's ability to carry out standard paper-and-pencil arithmetic (see, for example, Hembree & Dessart, 1986; 1992; Ruthven, 1996; Schools Curriculum and Assessment Authority, 1997), but it fails to take into account two important aspects of calculator use: the potential of the

calculator to be used as a powerful teaching aid in developing number sense; and the role of the calculator in redefining an appropriate curriculum, and how desirable mathematical performance should be assessed.

Pea (1985) distinguishes between the potential for technology to act as a *cognitive amplifier*—that is, to “change how effectively we do traditional tasks, amplifying or extending our capabilities”—and as a *cognitive re-organiser*—that is, as a tool whose use can “fundamentally restructure the functional system for thinking” (Pea, 1985, pp. 168, 170). According to Pea (1987, p. 99), the fundamental question is “How can technology support and promote thinking mathematically?”.

Acknowledging Pea’s distinction, Salomon, Perkins and Globerson (1991) further distinguish between the effects *with* technology use—that is, the effects while people are working with technology—and the effects *of* technology use—that is, all the “subsequent cognitive spin-off effects for learners working away from machines” (p. 2). According to Salomon, Perkins and Globerson (1991), intelligent technology—that is, technology able to “undertake significant cognitive processing on behalf of the user” (p. 3)—has the potential for the formation of an *intelligent partnership* where the division of labour between the human and the technology allows the “partnership ... [to] be far more ‘intelligent’ than the performance of the human alone” (p. 4). Jones (1996), referring to graphic calculators, proposes that when assessing mathematical intelligence, we need to consider whether it is the performance of a student alone or the partnership that needs to be assessed.

In the previous chapter, Sparrow and Swan focus on children forming intelligent partnerships with calculators by making a case for sensible *techno-literate* use of calculators. Ruthven, in the following chapter, discusses the long-term implications of the *Calculator-Aware Number* project (CAN) in terms of curriculum change and student learning outcomes.

This chapter draws on the findings of the *Calculators in Primary Mathematics* project to provide examples of how young children can form intelligent partnerships with calculators—using them as *cognitive amplifiers*—and elaborates on the subsequent effects in terms of children’s development of number sense and computational skills—their effects as *cognitive re-organisers*.

The Calculators in Primary Mathematics Project

The *Calculators in Primary Mathematics* project was a long-term investigation into the effects of calculator use on the learning and teaching of primary mathematics. The project, which commenced at Kindergarten and grade 1 in 1990, involved approximately 1000 children and 80 teachers in six schools in Melbourne, Australia. Children were given their own calculator to use whenever they wished in class. The project followed these children through to grade 3 and 4 in 1993, with new children joining the project each year as they started school.

The project was based on the premise that calculators, as well as acting as computational tools, are highly versatile teaching aids which can provide a mathematically rich environment for children to explore. One of the aims of the research was to document the extent to which teachers incorporated calculators into their teaching and the ways in which calculators were used. This was felt to be important because the project team believed that one reason why curriculum change has

been so slow is the fact that many teachers with a commitment to children's development of number concepts resort to the teaching of standard written algorithms because they see no other way to systematically involve children in activities related to number. Furthermore, teachers do not make appropriate use of the calculator as a teaching aid because they are unaware of how it can be used except as a computational tool or in very limited, short-term, classroom activities.

Project teachers were not provided with classroom activities or a program to follow. Instead they were regarded as part of the research team investigating the ways in which calculators could be used in their mathematics classes. Feedback and support were provided through regular classroom visits by members of the project team and through the sharing of activities and the discussion of issues at regular teacher meetings and in the project newsletter. As in the *Calculator-Aware Number* (CAN) project in the United Kingdom (Shuard, Walsh, Goodwin, & Worcester, 1991; Shuard, 1992), teachers devised, shared and adapted a wide range of activities, which were clearly superior to much of the published material available.

A large body of data was collected through an extensive program of classroom visits by the project team and by the teachers' self-reporting of activities. Over half of the teachers in the project were visited once a fortnight on average and all teachers completed approximately one record sheet per month. Four major ways of using the calculator emerged—as a recording device, as an aid to counting, as a computational tool and as an object to explore (Stacey, 1994; Stacey & Groves, 1996).

This chapter will not attempt to classify activities in this way, but will rather attempt to provide some illustrative examples of the extent to which the presence of the calculator encouraged teachers and children to explore beyond the range of numbers normally regarded as part of the curriculum and indicate the ways in which a deep intuitive understanding was being built up through continual—and often informal—exposure to such explorations of number. A more comprehensive account of teaching activities devised by project teachers can be found in Groves, Cheeseman, Dale and Dornau (1994).

Classroom Examples

Throughout the project, calculators were intended to be used alongside, not instead of, the usual classroom teaching aids such as counters, Unifix [stacking] cubes and base ten blocks. Like the calculator, these *concrete materials* can also be thought of as cognitive amplifiers—tools to extend children's capabilities to carry out tasks related to number. However, unlike calculators, the range of numbers which can be represented adequately with such materials is severely limited—quite small numbers for the materials typically used with young children and, even for base ten blocks with older children, only with some difficulty for numbers beyond the thousands or decimals.

The following examples illustrate how the calculator acted as a cognitive amplifier for project children by allowing them to access a much wider range of numbers than would be possible with only the use of traditional teaching aids. Often, this cognitive amplification took an apparently simple form, such as allowing very young children who have difficulty in writing numerals to readily record numbers or allowing young children to quickly carry out successive additions of a constant [skip counting]. At other times, the calculator was used as a tool to carry out computations which the children would not have been capable of doing otherwise (for example,

divisions such as $7 \div 4 = 1.75$)—nor perhaps even understanding without considerable reflection over a period of time.

The examples are also intended to illustrate the effect over time of children's reflections on such activities—that is, the effects of the calculator as a cognitive re-organiser. While it is possible that many children in classrooms which do not make use of calculators develop similar knowledge of numbers and the calculator only enables children to exhibit this knowledge, in at least some of the examples below there is a strong suggestion of the ways in which the use of the calculator has enhanced children's development of number concepts.

Large Numbers

One of the major ways in which the calculator was used, especially with younger children, was as a counting device, using the built-in constant function which allows counting by any chosen number, from any desired starting point. So, for example, keying in $1 + = = =$ results in the numerals 1, 2, 3 being displayed successively on the calculator. Similarly, keying in $1 + 3 = = =$ results in the numerals 4, 7, 10 being displayed, and keying in $1 - 2 = = =$ results in the numerals -1 , -3 , -5 being displayed. (Teachers buying calculators for use by young children should check that the calculator has such a built-in constant function for addition by carrying out tests like the examples given above, as not all simple four-function calculators have this capability and almost all scientific calculators require a more complicated procedure.)

One Kindergarten teacher initiated an activity, *number rolls*, which became popular with many project teachers. Long strips of paper were used to vertically record counting on by a constant. Many children began by counting by ones and continued to do so. Others, however, moved on to counting by numbers such as 5, 10 or 100. At least one child observed that counting by nines usually leads to the units digit decreasing by one each time, while the tens digit increases by one. By providing an easy means of generating data, the calculator encouraged children to look for patterns and many children spontaneously began to make conscious predictions about the next number in their sequence—even when they could not necessarily read the numbers aloud.

This, and other calculator counting activities, led many young children to exhibit a surprising facility with large numbers. The following examples are a very small sample of observations recorded in Kindergarten classrooms:

- Ben counted to 17 900 by 100s on his number roll. When asked what number would be reached after pressing equals two more times, he wrote 18 100, although he read it as eighteen hundred and one;
- Daniel wanted to show how he could count by 50s to 4000 on his calculator. Before he could be stopped, he had reached 64 250, which he was able to read aloud without hesitation—but later 102 350 presented him with problems;
- Christopher counted by 100s and said “I’m past 19 000 and up to 20 000. I’ve been doing it at home, 20 000 is one 2 and four zeros”; and
- Gavin knew that 50 000 was fifty thousand because “there are three zeros for 1000, and 50 has one zero, so fifty thousand must have four zeros.”

Perhaps the most striking recorded example of how the use of the calculator as a cognitive amplifier results in cognitive re-organisation is one shown on the videotape *Young Children Using Calculators*. (Groves & Cheeseman, 1993). Children in a Kindergarten class were told to find a partner and “go away and play” with their calculator, and later in the lesson they would come back and share what they had done. While the children were “playing” the teacher moved around the class, engaging children in often quite lengthy discussions.

Teacher: Can you tell me what you have been doing.

Simon: Well I pressed one, zero, zero, zero, zero, zero, zero.

Teacher: What number is that?

Simon: A million.

Teacher: How do you know it's a million?

Simon: Because it has six 0's. My mum told me a million has six 0's.
[Simon starts counting by ones and reaches 1 000 009]

Teacher: What do you think the next number will be?

Simon: A million and ten.

Teacher: [To Simon's partner] Let's give him a number ... What will we give him to stop at?

Alex: A million and five.

Teacher: He's gone way past that ... What's larger? ...

Alex: One million one hundred.

Simon: I think there's no such thing as one million one hundred.

Teacher: You don't think there is such a number ... all right, let's see. Just stop now ... what's your number now? [1 000 079]

You are going to stop at one million one hundred — if you think it's there. Stop. What's your number now?

Simon: One million and ninety-four.

Teacher: Now what are you looking for?

Simon: One million one hundred.

Teacher: Now what's that?

Simon: I think I'll get it. [1 000 102]

Teacher: So what did you do?

Simon: I went past it!

Teacher: So do you think there is such a number as one million one hundred now?

Simon: [Nods].

While Simon clearly had an excellent grasp of counting and number recognition for a child in kindergarten, and could possibly have been able to carry out a similar activity without a calculator, it is extremely unlikely that he would have done so. The calculator amplified and extended Simon's capabilities by providing a dynamic display of the result of Simon's desired actions—counting by ones from a million—without him needing to painfully record each number. This in turn allowed Simon to focus on

thinking about the results and restructure his knowledge of large numbers to include not only those in the range of 1 000 000 to 1 000 099, but numbers beyond 1 000 100.

Of course there were many children in Kindergarten classes who still had difficulty recognising numerals and were hesitant about all aspects of using the calculator. One of the overwhelming responses by teachers to the introduction of calculators was to express surprise at the range of understanding of number exhibited by children in their classes. Many teachers said that not only had they previously been unaware of the extraordinarily wide range in their classes, but that they had always based their teaching on a predetermined curriculum, without taking into account the extent to which class activities were at an appropriate level for the children. Teachers viewed this as a serious challenge to their teaching.

Negative Numbers

By using the constant function to count backwards, children discovered and explored negative numbers. In a Kindergarten class where children had been discussing and drawing “What lives underground?” Alistair said “Minus means you are going underground,” and wrote some negative numbers. When questioned what would be the first number above the ground, he said “zero”. “Underground numbers” were used and discussed freely in many classrooms. Kylie’s illustration is shown in Figure 1.

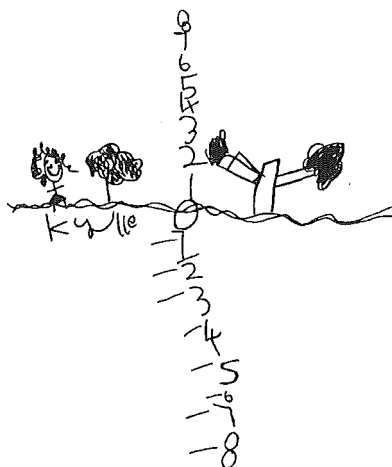


Figure 1: Kylie’s underground numbers

These were by no means isolated instances—many young children developed a quite sophisticated understanding of the meaning of negative numbers, as illustrated by the examples below:

- Jason, in Kindergarten, keyed in $10 - 1 = = = \dots$ and said “When it gets to 0 it counts up again by 1 but it has a minus sign”;
- Jamie, also in Kindergarten, said “The smallest number I know is -3095 and the biggest is 3099 ”;

- John, in grade 1, said “When you’ve got negative numbers like -98 and -99 , you might think -99 is bigger, but it’s not because -99 is further away from zero, so it’s smaller. Everything is opposite”;
- Grade 1 and 2 children were trying to make the smallest number for a maze. The teacher asked “Is there a smaller number than zero?” Jane replied “Negative 5, negative 20.” Lucy said “Negative infinity” and when the teacher asked what this meant she replied “It never ends”;
- Grade 2 children’s responses to “What does -5 mean?” included Fletcher: “Take away 5”; Sarah: “It goes underground 1, underground 2, underground 3, ... I call them underground because it’s underground”; Cameron: “ -5 means 5 under zero”; Jessica: “It goes $-1, -2, \dots$ It’s not take away anymore—it goes forward”.

Responses like these arose in many different situations. As well as counting activities, teachers often used story books as a basis for mathematical activities, while other teachers invented uses for the calculator as a recording device—based on children’s spontaneous use of their calculator as a “scratch pad” to record things such as their phone numbers, the date and number sequences like 12345678. For example, one grade 1 and 2 class played a game *number line-up* where children were asked to enter a number less than 100 on their calculator and then, starting with a small group, order themselves from smallest to largest number. More and more children would join the line-up until the whole class was in order. After playing this game a few times, some children spontaneously started using negative numbers and were able to correctly put themselves in order.

Decimals

Many instances of young children working with decimals—often spontaneously—were recorded. Some children were confronted by decimals almost immediately, as is indicated by the drawing made by a Kindergarten child on the first day that calculators were used in class, two weeks after the school year commenced (see Figure 2).



Figure 2: Kindergarten child’s drawing on the first day of using calculators

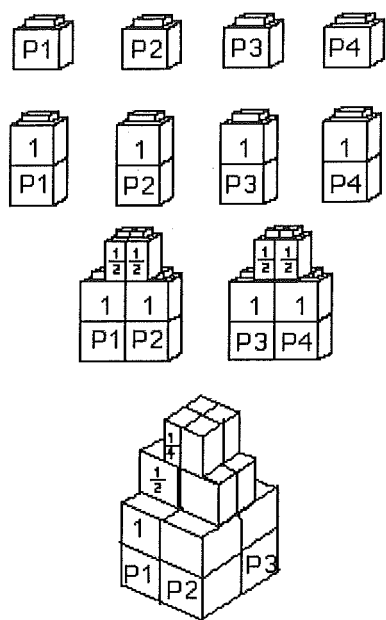
Decimals occurred in a variety of contexts, with the two most common occurring when children who were presented with real world problems involving sharing used their calculators to perform a division and obtained a decimal answer; and when children spontaneously decided to count by a decimal such as 0.1.

For example, some Kindergarten children wanted to share 55 cookies among their 10 teddy bears. One child had discovered the “sharing sign” on the calculator and had informed classmates of her discovery. Two children got the answer 5.5 on their calculator. The teacher discussed the idea that 0.5 was a half and one child commented “Oh, that’s five and a half cookies then.” She remembered this and explained it to a member of the project team who visited the classroom some time later, repeating her calculation of $55 \div 10 = 5.5$. Figure 3 shows the ten bears that Zoe and Julianne pasted onto their paper. They had drawn five and a bit cookies alongside each bear.

$$55 \div 10 = 5.5$$



Figure 3: Sharing cookies among teddies



$$1 + \frac{1}{2} + \frac{1}{4} = 1 + 0.5 + 0.25$$

Figure 4: David and Brodie sharing seven things between four people

Grade 1 children were asked how to share seven things between four people. Some children said it could not be done. Brodie said her calculator showed that 7 shared between 4 was 1.75. The teacher asked if anyone could show how to share seven between four in another way. David said: “Yes. You take four Unifix [stacking cubes] and that can be the four people. Then these [seven more Unifix] can be shared like this.” He put one Unifix cube on each person and said: “That’s one for each person and I have three left. If I push two people together and put a block on top, they can have half a block each. So each person has one block and a half and a quarter.” Through general discussion, children decided that a half was two quarters, so each person had one and three quarters. Brodie remembered that they had discovered a few days earlier that four lots of 0.25 make 1, so each person had one and three quarters—which was 0.75 on her calculator. It was finally agreed that seven could be shared between four people, with each person getting one and three quarters, which was the same as 1.75 (see Figure 4).

Grade 2 children encountered decimals while discussing how to make a pictograph of the results of a tree survey. A group of 7 children needed to cut out 64 pictures of trees to paste onto their chart. When asked by the teacher how many trees each child in the group would need to make, some children spontaneously used their calculators and found that $64 \div 7 = 9.1428571$. The teacher wrote down their answer and asked what it meant. A child quickly replied: “It is nine and a bit. So if we made ten each we would have some left over—actually we would have six left over.”

Other grade 2 children made dinosaur ‘footprints’, choosing their own starting and finishing points and filling in the steps to get from the start to the finish. Many children used decimals and negative numbers in the process (see Figure 5).

$$-4.3 + 4.3 + 10.43 = 11$$

$$100.9 + 0.2 + 99.9 + 0.9 = 2000$$

$$1.3 + 0.8 + 2.3 + 1.3 = 4.1$$

$$3.2 + 2.8 + 3 = 9$$

$$2000 + 91 + 2000 + 91 + 2000$$

Figure 5: Dinosaur steps

Some other examples of the use of decimals are given below.

- Grade 1 children tried to share 30 sweets among eight people. Julianne got 3.75 on her calculator and said “that means three and three quarters” and drew ● ● ● ● ;
- Grade 2 students were asked, “How many ways can you get 30 legs using chickens and dogs?” Nicolas said “7.5 dogs because 0.5 is a half”; and

Grade 3 and 4 children were asked to start with 0.1 and experiment until they got a whole number. Eva used $0.1 \times 40 = 4$. Patrick found $0.1 \times 10 = 1$, while Keith used $0.1 \times 1000 = 100$. Bass entered $0.1 + = = = = \dots$. Peter was very excited as he could see a pattern building up from $0.1 \times 20 = 2$, $0.1 \times 30 = 3$, $0.1 \times 40 = 4$, $0.1 \times 50 = 5$.

One grade 2 teacher commented that she probably would never have dealt with decimals if she had not been reading about another teacher's experience in the newsletter, or without calculators. It is probably not surprising that the major difference found between children who had long-term experience of calculator use and those who did not was in the area of recognition of decimals and the understanding of their meaning.

Learning Outcomes

While learning outcomes are not a focus for this chapter, a brief summary of the findings are included below because of their perceived link with changes in teachers' beliefs and practice.

Classroom observation confirmed the hypothesis that some children were developing concepts related to large numbers, negatives and decimals, at an earlier age than expected. In an effort to discover the extent to which children across the whole spectrum were able to engage in these exciting discoveries, interviews were conducted in 1991 with a 20% random sample of children at each of Kindergarten, grade 1 and grade 2 at two of the six schools. These interviews were suggested by the project teachers. All three interviews included questions where the children were shown 'large numbers' on flashcards and asked: "Can you tell me what this number is?"

A significant proportion of the children were able to read numbers which are usually considered well beyond the curriculum at each of the grade levels, as now specified by the Victorian *Curriculum and Standards Framework II: Mathematics* (Board of Studies, 2000). For example, almost half the Kindergarten children could correctly read 74, while the same percentage of grade 1 children were able to read 1435, and 65% of grade 2 children could read 3294. In addition, 30% of grade 2 children (the only grade level asked) were able to correctly explain the meaning of -5 as "5 below zero" or similar. However, while over a third of grade 2 children could read 5.7 correctly, only 9% when asked, "How big is 5.7?" were able to give a correct answer, such as "a bit bigger than 5". Thus, many children showed some understanding of negative numbers and many could also correctly read a decimal, but relatively few could give an indication of its size—for example, by comparing it with a whole number.

An extensive program of testing and interviews, with and without calculators, was conducted at the grade 3 and 4 levels from 1991 to 1993, using the last cohort of children at each year level who had not taken part in the project as the control group. A total of approximately 1500 children were given a written test and a test of calculator use, while over 10% of these children also took part in one of two 25-minute interviews. Children with long-term experience of calculators, while performing equally well on simple calculator tasks as children without such experience, performed significantly better on those tasks which required some knowledge of negative numbers and decimals. These children also showed somewhat better understanding of the number system and were somewhat better able to identify an appropriate operation in a series of word problems. At interview, they performed better overall on both sets of computation

items than children without such experience—in one set they could use any tool of their choice, while the other required mental computation only. They also performed better on a range of computation and estimation tasks and some ‘real world’ problems; exhibited better knowledge of number—particularly place value, decimals and negative numbers; made more appropriate choices of calculating device; and were better able to interpret their answers when using calculators, especially where knowledge of decimal notation or large numbers was required.

Despite fears expressed by some parents, there was no evidence that children became reliant on calculators at the expense of their ability to use other forms of computation. No detrimental effects were observed in either the interviews or the written tests. For further details regarding the effect of calculators on learning outcomes, see Groves (1993; 1994a; 1994b); Stacey (1994a; 1994b); Stacey and Groves (1994).

Conclusion

This chapter has described a range of ways in which children and teachers in the *Calculators in Primary Mathematics* project used calculators as a powerful teaching aid to facilitate children’s exploration of number. Results from an extensive program of testing and interview confirmed that children with long-term experience of calculators performed better than children without such experience on a range of tasks. In particular, they exhibited a better knowledge of number—particularly in relation to place value, decimals and negative numbers—and made more appropriate choices of calculating device. Young children were clearly able to form *intelligent partnerships* with calculators, using them as *cognitive amplifiers* to extend the range of tasks they were able to tackle; while the calculator appeared to act as a *cognitive re-organiser* which enabled children to extend and restructure their knowledge of number well beyond what would be normally expected.

The extensive results from this project confirm other research evidence that calculators can be used in highly productive ways with young children and that their use, rather than undermining learning, enhances children’s number sense.

References

- Board of Studies (2000). *Curriculum and standards framework II: Mathematics*. Melbourne: Victoria: Board of Studies.
- Dick, T. (1988). The continuing calculator controversy. *Arithmetic Teacher*, 35(8), 37–41.
- Duffin, J. (2000). Invited review: We must not let the future be the prisoner of yesterday’s ideas. *Teaching Mathematics and Its Applications*, 19(2), 56–61.
- Gelernter, D. (1998). Put down that calculator, stupid! *New York Post*. May 21, 1998.
- Groves, S. (1993). The effect of calculator use on third graders’ solutions of real world division and multiplication problems. In I. Hirabayashi, N. Nohda, K. Shigematsu & Fou-Lai Lin (Eds.), *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education*. (Vol. II, pp. 9–16). Tsukuba, Ibaraki, Japan: University of Tsukuba.

- Groves, S. (1994a, April). *Calculators: A learning environment to promote number sense*. Paper presented at the American Educational Association Research Association 1994 Annual Meeting, New Orleans. (ERIC Document Reproduction Service No. ED 373 969.)
- Groves, S. (1994b). The effect of calculator use on third and fourth graders' computation and choice of calculating device. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education*. (Vol. III, pp. 33–40). Lisbon, Portugal: University of Lisbon.
- Groves, S., Cheeseman, J., Dale, J. & Dornau, A. (Eds.) (1994). *Calculators in primary mathematics: The newsletters*. Geelong: Centre for Studies in Mathematics, Science and Environmental Education, Deakin University.
- Hembree, A. & Dessart, D. (1986). Effects of hand-held calculators in pre-college mathematics education: A meta-analysis. *Journal for Research in Mathematics Education*, 17, 83–99.
- Hembree, A. & Dessart, D. (1992). Research on calculators in mathematics education. In J.T. Fey & C.R. Hirsch (Eds.), *Calculators in mathematics education: 1992 yearbook* (pp. 23–32). Reston, VA: NCTM.
- Jones, P. L. (1996). Handheld technology and mathematics: Towards the intelligent partnership. In P. Gomez & B. Waits (Eds.), *Roles of calculators in the classroom*. (pp. 87–96). <http://ued.uniandes.edu.co/roless-calc.html> [Accessed 17 May 2001]
- Klein, D. (2000). Math problems: Why the Department of Education's recommended math programs don't add up. *American School Board Journal*, 187(4), 52–57.
- Pea, R. D. (1985). Beyond amplification: Using the computer to reorganise mental functioning. *Educational Psychologist*, 20(4), 167–182.
- Pea, R.D. (1987). Cognitive technologies for mathematics education. In A. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 89–122). Hillsdale, NJ: Lawrence Erlbaum.
- Ralston, A. (1999). Let's abolish pencil-and-paper arithmetic. *Mathematics Education Dialogue*, 2(3), 2.
- Ruthven, K. (1996). Calculators in the mathematics curriculum: The scope of personal computational technology. In A. Bishop et al. (Eds.), *International handbook of mathematics education* (pp. 435–468). Dordrecht: Kluwer Academic Publishers.
- Salomon, G., Perkins, D.N. & Globerson, T. (1991). Partners in cognition: Extending human intelligence with intelligent technologies. *Educational Researcher*, 20(3), 2–9.
- Schools Curriculum and Assessment Authority (1997). *Discussion paper No. 9. The use of calculators at key Stages 1–3*. London: SCAA.
- Shuard, H., Walsh, A., Goodwin, J. & Worcester, V. (1991). *Calculators, children and mathematics*. London: Simon & Schuster.
- Shuard, H. (1992). CAN: Calculator use in the primary grades in England and Wales. In J. T. Fey & C. R. Hirsch (Eds.), *Calculators in mathematics education* (pp. 33–45). Reston, VA: National Council of Teachers of Mathematics.

- Stacey, K. (1994a). *Calculators in primary mathematics: An analysis of classroom activities*. Paper presented at the American Educational Association Research Association 1994 Annual Meeting, New Orleans. (ERIC Document Reproduction Service No. ED 373 968.)
- Stacey, K. (1994b). Arithmetic with a calculator: What do children need to learn? In G. Bell, B. Wright, N. Leeson & J. Geake (Eds.), *Challenges in mathematics education: Constraints on construction* Proceedings of the Seventeenth Annual Conference of the Mathematics Education Research Group of Australasia (pp. 563–570). Lismore: Mathematics Education Research Group of Australasia.
- Stacey, K. & Groves, S. (1994). *Calculators in Primary Mathematics*. Paper presented at the Research Precession of the 72nd Annual Meeting of the National Council of Teachers of Mathematics, Indianapolis. (ERIC Document Reproduction Service No. ED 373 963.)
- Stacey, K. & Groves, S. (1996). Redefining early number concepts through calculator use. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning: A research monograph of MERGA/AAMT* (pp. 205–225). Adelaide: Australian Association of Mathematics Teachers.

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Building a Calculator-Aware Number Curriculum: The CAN Project and Beyond

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The pioneering *Calculator-Aware Number (CAN)* project took place between 1986 and 1989. The project team worked in collaboration with teachers in four clusters of primary schools in different regions of England and Wales to develop a new approach to the teaching of number, based on the following principles (Shuard, Walsh, Goodwin & Worcester, 1991, p. 7):

- classroom activities should be practical and investigational, emphasising the development and use of language, and ranging across the whole curriculum;
- encouragement should be given to exploring and investigating 'how numbers work';
- the importance of mental calculation should be emphasised, and children should be encouraged to share their methods with others;
- children should always have a calculator available, and the choice as to whether to use it should be the child's not the teacher's; and
- traditional written methods of calculation should not be taught, and children should use a calculator for those calculations which they could not do mentally.

In what senses, then, was this a 'calculator-aware' number curriculum? First, as the last two principles indicate, it acknowledged the way in which electronic means of calculation were rapidly displacing written methods outside school. Correspondingly, within project schools, electronic calculators replaced written methods as the computational means of secure resort. Given the degree to which the established number curriculum centred on the development of standard written methods of calculation, this was a radical change, not simply removing content and releasing time, but removing a major organising strand of the curriculum. Equally—as the first two principles indicate—this new curriculum recognised that numeracy involves exercising number sense as well as affecting numeric calculation. Here, the curriculum was 'calculator-aware' in the further sense of exploiting use of calculators to stimulate and support children's exploration of properties of number. Finally, these varied concerns were drawn together—as the middle principle indicates—around a curricular strand which focused on comparing and refining children's strategies of both mental and informal written calculation, seen as a means by which number concepts could be actively developed. This, then, was a 'calculator-aware' number curriculum, but not a 'calculator-based' one. Indeed, in some senses, the calculator might be said to have

acted more as catalyst than central agent: underwriting the shift in emphasis from written to mental calculation, and facilitating the shift from a pedagogy of instruction towards a pedagogy of investigation.

Analysis of the range of experiences reported from the CAN project suggests that the active contribution which calculators made can be understood in terms of four ideal types of use, which will be illustrated and examined in turn: implementing, checking, tinkering and modelling. *Implementing* is using a calculator to carry out an already formulated calculation. Such use of calculators enabled children in the CAN project to carry out—and to envisage carrying out—calculations which would not otherwise have been feasible for them; for example, single calculations involving large numbers, or multiple calculations, notably those arising from authentic problem situations or in investigating number patterns. *Checking* is using a calculator to review a calculation already carried out. Where this calculation has been done mentally, the calculator is a means by which a pupil can gain rapid feedback on the result, and so indirectly on the mental strategy employed. Checking, then, has the potential to support children's monitoring of their mental strategies, not simply the results of their calculations. Second, where a calculation has been executed on the calculator, there are important alternatives to simply repeating the same calculation. One is to 'reverse' the calculation, with a view to working back from the result to the original data—for example, where a multiplication has been carried out, applying the inverse division. Another alternative is to reformulate the calculation—for example, repeated addition as multiplication. Checking, then, has the potential to become a setting for learning about the equivalence of variant calculations, so developing and refining ideas about the structure of the number system and number operations.

By reducing the 'effort' of calculation, calculators encourage a predisposition towards more spontaneous and speculative calculation. *Tinkering* is using the calculator to solve a problem by experimenting with some scheme of calculation until an acceptable solution is found. A prominent example is the general strategy of 'trial and improvement' which became widely used in the CAN project. In this strategy, a speculative solution is proposed to a problem—by guessing or estimating, which is then tested, through some appropriate scheme of calculation—against the condition it must satisfy. Then, in the light of the information that the user is able to extract from this feedback—about the nature of the 'gap' between the actual result and the one sought—the proposed solution is revised and retested. This cycle continues until a satisfactory solution is reached. An example from the early primary years is of a child trying to find what number to subtract from 67 in order to arrive at 18; estimating 40 and computing $67 - 40$ on the calculator to get 27; then revising the estimate upwards (Shuard et al., 1991, p. 71).

Modelling is using the calculator to effect calculations with the intention of exemplifying some aspect of the operation of number and calculation so as to support learning about it. For example, in response to an early primary pupil who has written two thousand and ten as 200010, a teacher uses a calculator to show her $2000 + 6 = 2006$ and $2000 + 13 = 2013$; the pupil herself then carries out further addition calculations, such as $2000 + 17 = 2017$, before progressing to entering numbers directly—such as 2039 (Shuard et al., 1991, p. 13). Similarly, in response to the calculation $1 \div 4 = 0.25$, a pupil speculates that 'a quarter is 0.25', and then follows this up with $0.25 + 0.25 + 0.25 + 0.25 = 1$ and then $0.25 \times 4 = 1$ (Shuard et al., 1991, p. 21). There is a sense, of course, in which this is a form of checking. However, what distinguishes modelling from the other ideal types is the guiding concern with seeking

meaning in, and establishing knowledge of, number and calculation. It is not the calculation itself which is of primary interest, but the mathematical ideas, principles and processes underlying it, as embodied in the operation of the calculator.

The tangible outcomes of the CAN project are recorded in a text and video prepared by the project team (Shuard et al., 1991). These illustrate the curriculum principles presented above through a collage of classroom activities and accounts. The project team were able to draw on examples from the earlier years of primary school, but not the later years as the cohorts of children involved in the project had not yet reached that stage. A more structured curriculum plan was not developed. This reflected the pedagogical approach embraced within the project.

The teachers began to develop an exploratory and investigative style of working, which allowed the children freedom to take responsibility for their own learning. Topics for exploration took the place of practice exercises as the prevailing classroom activities. Because the number sections of the mathematics schemes used in the schools had been discarded, the teachers were able to move towards a different style of working. No longer did they have to 'cover' set topics in a set order. They began to notice that children's mathematics learning did not seem to progress in the ordered linear way in which it was traditionally structured. Individual children seemed to be putting together the network of mathematical concepts in their own individual ways (Shuard, et al., 1991, p. 44).

The project team reported very favourable findings from one of the participating clusters of schools in which the mathematical achievement of the first cohort of project pupils was compared with that of peers in other schools (Shuard et al., 1991, pp. 59-60). The following year, the second cohort was involved in a similar comparison with results still favourable to the project pupils, but less markedly so (Foxman, 1996, p. 47). Of course, the tests used did not seek to assess facility with the written methods of calculation which had been a major focus of the mathematics curriculum in the comparison schools. Indeed, as the project team pointed out, for a range of reasons "it would not be possible to equate the conditions in project schools and control schools, and this kind of quantitative evaluation might be misleading" (Shuard et al., 1991, p. 55).

The Evolution and Long-term Impact of CAN

As the original CAN project drew to a close in the summer of 1989, a new national curriculum came into force. A research study examined the experience of a cohort of pupils who entered reception class (R) during the 1989/90 school year, progressing to the final year (Y6) of primary education in 1995/96. During that final year, data was gathered in six neighbouring primary schools. Three of these schools had participated in the CAN project between 1986 and 1989, and then in the much smaller-scale continuation project from 1989 to 1992. According to teachers in these schools, the major influence of the introduction of a national curriculum and subsequently of national assessment had been threefold (Ruthven, 2001). First, some of the expansiveness of investigative work had gone, and there was a stronger tendency to structure and foreclose an activity than in the past. Second, although calculators continued to be readily available in the classroom, there were occasions when their use was challenged or proscribed. Third, standard written methods of recording and calculating had been reintroduced. In the lower primary school, teachers felt obliged to introduce pupils to vertical methods of recording, and to 'sums' presented in this way.

In the upper primary, standard written methods were more prominent, although the expectations of secondary schools were often cited as the direct reason for this. Generally, however, the tenor of teachers' accounts was of seeking to retain valued principles and activities from CAN; to establish the legitimacy of these principles and activities within the new order; and to tighten aspects of their implementation.

Nevertheless, the teachers also pointed to pedagogical tensions arising within CAN. They had developed a more subtle view of the complexities of supporting pupils' development of methods of calculation. They were conscious of having to manage an important tension between personal insight and authenticity on the one hand, and accuracy and efficiency on the other.

We've built on what the children have actually used ... try out the different methods and encourage them to find the one they feel most happy with ... There is one child I did change ... because he was not accurate, and he was slow. His methods were so long-winded... It is important that children do have quick, accurate methods. One of the things which is really important is... that the children have conceptual awareness of what's happening with the numbers. If they know that then they are secure. But some of the children are going through the motions with methods they don't understand. [Richard]

We put [pupils' strategies] very high up [but] the older a child is the more likely I am to say, 'that's fine but it takes twice as long as this one' There is kind of a seductiveness in working investigatively and they forget that there can be a directness that is important as well. [Stephanie]

Another issue which emerged from teachers' accounts related to the systematisation of CAN within schools. Salient themes here were of the uncertainty and effort arising from the abandonment of a conventional mathematics scheme, with limited alternative means of support.

I came to this school having a fairly sketchy knowledge of CAN, having seen it in operation, but having a sketchy knowledge about how to proceed, and finding no resources. The resources there were photocopyable resources and packs. There would be one copy so you had to have copies made. It was incredibly hard work preparing lessons each day. [Richard]

We more or less abandoned schemes and went in at the deep end with CAN. Two members of staff in particular were heavily involved with it and went to meetings and then fed back to staff. But, as I remember, you were left floating about a bit and not knowing what was right or what was wrong to do. I remember thinking if I just give them investigations and problems and help them to solve them, that's how I'll survive this. You felt as though there was nothing to support you... When you have a scheme, you don't use it rigidly, but you know it is there as a support for you if you need it... The two who went to the meetings seemed to be more capable at it. You needed to go to the meetings. They got the ideas from the meetings. We just got the 'trickle down'. [Tricia]

In these circumstances, it was difficult to plan for continuity and progression in children's learning, both from lesson to lesson and from year to year.

In CAN it was difficult to know how to progress. After an exciting lesson you thought 'Where do I go now? Where do I take them next?' You'd be rooting around for ideas. [Tricia]

There was no structure through the school ... I noticed in my first year that teachers were photocopying an investigation for Year 3 children and the same one was being used for Year 6 children, and nobody knowing what the children had covered at all. [Richard]

One important effect of the national curriculum and assessment reforms had been to press schools to develop a more systematic approach to number, building on the national frameworks.

The research study compared the progress of pupils in the post-project schools with that of their peers in the non-project schools (Ruthven, Rousham, & Chaplin, 1997). National assessment levels awarded at ages 7 and 11 were analysed to determine whether the odds of high or low attainment in mathematics differed between schools, after taking account of the general scholastic attainment of pupils. At age 7, the odds of high mathematics attainment were found to be significantly greater in the post-project schools, as also were the odds of low mathematics attainment. In the post-project schools, then, pupils were more likely to be found at either extreme of the attainment distribution. Comments from the teachers of the cohort in post-project schools suggest that a plausible explanation is that the emphasis on investigative and problem-solving tasks within CAN produced a greater differentiation of experience between pupils, creating higher expectations of, and greater challenges for, successful pupils, but providing less systematic teacher intervention to structure and support the learning of pupils who were making poor progress.

One of the things that keeps me working in this way is that low ability children don't get so complexed about it ... I think the weak ones do benefit from a lot of talk and being involved in things. They are not excluded because they didn't manage to get quite as much done. And for the high flyers, I think it is a brilliant way of working because they can go as far as they want; there is no ceiling on them. They can take off and go a long way with things and the talk is good for them at that end. [Stella]

You always thought: "Do children really understand—particularly the less able children? Do they really understand what it is they are doing?". I think it showed up with more able children, if they got an answer which was clearly wrong, they knew it was wrong. But that estimating thing was not there with less able. You'd have outrageous answers and they wouldn't have a clue it was not right ... I didn't ensure that, like I do now, that children could add up quickly, mentally in their head... Looking back I think I should have done that. That would have helped the less able with their estimating ... Some children struggled, but the children who had a gift for maths did very well. If they had a good understanding of the structure of numbers and estimating skills, then they went quite far. [Tricia]

However, this differential pattern did not persist through to the results at age 11 where no substantial differences were found between non-project and post-project schools, either on national assessments or on specially devised measures focusing on a range of number concepts. Similarly, no differences were found in reported enjoyment of number work. However, there were clearly discernible trends for pupils in the post-project schools to rate mental calculation more positively than pupils in non-project schools.

Analysis of the strategies used by pupils in tackling a set of number problems strengthened these findings (Ruthven, 1998). Pupils in post-project schools proved more prone to calculate mentally. Whereas 38% of pupils in post-project schools tackled all the problems mentally without any use of written or calculator computation, only 19% of pupils in non-project schools did so. While only 24% of pupils in post-project schools used written or calculator computation on more than one occasion, 52% of pupils in non-project schools did so. Pupils in post-project schools also proved more liable to adopt relatively powerful and efficient strategies of mental calculation. For

example, in response to the problem of calculating the cost of five 19p stamps, the more powerful mental strategies involved distribution—‘Times ten is fifty. Times nine is forty-five. Add them together.’—and sometimes compensation—‘Five twenties which was one pound. Then I took away five’. Whereas 55% of pupils in the post-project schools used a mental strategy of this type to tackle the problem, only 22% of those in the non-project schools did so.

It is plausible to see these outcomes as reflecting the contrasting numeracy cultures of the two groups of schools. In the post-project schools, pupils had been encouraged to develop and refine informal methods of mental calculation from an early age; they had been explicitly taught mental methods based on ‘smashing up’ or ‘breaking down’ numbers into component parts; and they had been expected to behave responsibly in regulating their use of calculators to complement these mental methods. In the non-project schools, daily experience of ‘quick-fire calculation’ had offered pupils a model of mental calculation as something to be done quickly or abandoned; explicit teaching of calculation had emphasised approved written methods; and pupils had little experience of regulating their own use of calculators.

Calculator-based Computational Strategies

The original CAN project put into action the idea that: “With mental methods... as the principal means for doing simple calculations ... calculators ... are the sensible tool for difficult calculations, the ideal complement to mental arithmetic” (Plunkett, 1979, p. 5). However, as noted in the previous section, the introduction of a national curriculum led to a significant weakening of this position in CAN schools, accentuated by components of national assessment which framed problems in terms of standard written methods, or required pupils to show written working, or barred use of calculators. Consequently, the experience of the pupil cohort examined in this follow-up study had not placed such a strong emphasis on developing pupils’ expertise in using calculators. This was illuminated by the analysis of pupils’ responses to a realistic number problem (Ruthven & Chaplin, 1997).

The ‘coach problem’ was a close variant of one featured as an example in the national curriculum: *313 people are going on a coach trip. Each coach can carry up to 42 passengers. How many coaches will be needed? How many spare places will be left on the coaches.* Pupils were told that they could work out the problem however they liked; using their head, pen and paper, or calculator, or a mixture of these. The patterns of response by pupils in post-project and non-project schools were not dissimilar, with around 60% making some use of a calculator, and three broad types of calculator-based strategy in evidence: direct-division, repeated-addition, and trial-multiplication. Each of these gave insights into forms of expertise which pupils need to develop in order to make effective use of a calculator.

The most common use of a calculator was for direct division. The responses of Karen and Damon (Figure 1) exemplify features of such responses which were widespread. It seems that Karen’s initial interpretation of the string of digits on the calculator display is that she has miss-keyed; and so she checks by re-keying. When this produces the same result, it appears that her next interpretation is that she has entered the numbers in the wrong order within the calculation, and her checking shifts towards tinkering. Such responses reflected an expectation—or perhaps an aspiration—that the

<i>Karen's Response</i>	<i>Damon's Response</i>
Karen keys [313][÷][42][=]7.452380952	Damon keys [313][÷][42][=]7.452380952
Karen: Whoopsee!	Interviewer: What have you got? Any good?
Interviewer: What have you got?	Damon: About seven coaches.
Karen: I've got loads of numbers.	Interviewer: About seven coaches.
Interviewer: Are they any good to you?	[pause]
Karen: No	Damon: I think it's four.
Interviewer: Why?	Interviewer: Four.
Karen: I don't know	Damon: Yeah.
Interviewer: Can you understand what they say?	Interviewer: Spare places?
Karen shakes her head	Damon: Yeah.
Interviewer: Okay.	Interviewer: How did you work that bit out?
[pause]	Damon: Because it's seven point four.
Karen re-keys [313][÷][42][=]7.452380952	
[pause]	
Karen keys [42][÷][313][=]0.1341853035	

Figure 1: Calculator-based direct-division strategies for the coach problem

result of a division should be a whole number. It is not only that the commonsense of the problem points in this direction. Pupils' experience of mental and written division had been as a process within the system of whole numbers, yielding a quotient and possibly a remainder; whereas the calculator treats division as a process within the extended number system incorporating decimals. Equally, pupils' contact with decimals had been predominantly in terms of money and measures. Karen did not recognise the string of digits as incorporating a decimal resulting from an 'inexact' division. And although Damon did recognise this form of result, his interpretation of the fractional part was in terms of a remainder. These examples highlight the special character of calculator division and the demands that it makes on pupils' mathematical understanding. Indeed, carefully designed calculator-based tasks can support development of pupils' understanding of relationships between division, fraction and decimal concepts—for example, by investigating which division calculations produce a particular decimal part; or by exploring the way in which a calculator rounds repeating decimals resulting from division calculations (van den Brink, 1993).

<i>Liam's Response</i>	<i>Kath's Response</i>
Liam: So you need to add up how many forty-twos go into. I'll do that. I'm sure you could do it a quicker way but, well.	Kath: 42 times
Liam keys [42][+][42][+][42][+][42][+][42][+][42][+] monitoring intermediate totals	Kath keys [42][x][=]1764
Liam keys [252][+]	Kath re-keys [42][x][=]1764
Liam: Oh no!	Kath: I thought if you could do forty-two, times and then equals, it should keep going, forty two, eighty four like that and say how many forty- twos to get up to that.
Interviewer: Where have you got to? What's happened?	
Liam: Hmmm. Don't know.	

Figure 2: Calculator-based repeated-addition strategies for the coach problem

Another use of the calculator was for repeated addition. The example of Liam (Figure 2) is typical, both in its keying pattern and in its eventual breakdown. The calculator leaves no trace of intermediate results, making any extended calculation incorporating a parallel mental computation extremely vulnerable to failure through miss-keying or losing track of where the calculation has reached. Pupils who tried to compute mentally without recording had similar difficulties. Whether working wholly mentally or with the calculator, maintaining some form of written record provides an important means of augmenting working memory. Alternatively, use of the calculator constant function offers a way of simplifying and expediting repeated computations of this type. Kath was the only pupil who attempted this (Figure 2). She knew that she wanted to repeat an operation, she knew how to get the calculator to do that, she knew that she wanted the multiples of 42, but misconstrued this as a matter of repeated multiplication rather than repeated addition.

A final use of the calculator was for trial multiplication, normally taking an estimate of 7 from direct division and keying [42][x][7][=]294, and often then calculating—usually mentally—that 294 is 19 short of 313. However, the typical interpretation of these findings was that 7 coaches were required with 19 spare places—reflecting a misconceived association between ‘remainder’ in the calculation and places ‘left’ in the problem. The only successful use of trial multiplication by Joanne (Figure 3) took a rather different form, since she embarked on it immediately as her opening strategy, rather than following on from direct division. Using the machine to carry out computations in a predictably routine way, Joanne freed her attention to monitor her strategy and interpret results. And this devolution of computation was systematic, even extending to multiplying 42 by 10—something which Joanne was very capable of doing mentally (earlier in the interview she had successfully mentally multiplied 24 by 10, answering within one second). Nicki (Figure 3) also used a trial-and-improvement strategy from the start, similarly devolving calculation to the machine. This enabled her to work with an unusual representation of the problem, in which she focused on the

average number of passengers per coach, employing trial division. This example also brings out another important feature of trailing strategies—that they are disposed to be self-correcting. Nicki’s misreading of 62 is not critical because it is quickly superseded by the next trial.

<i>Joanne’s Response</i>	<i>Nicki’s Response</i>
Joanne keys [42][x][12][=]504	Nicki keys [313][÷][5][=]62.6
Interviewer: Why did you do that?	Nicki: Fifty-two.
Joanne: Forty-two times any number, but it was a bit too high.	Nicki keys [313][÷][7][=]44.71428571
Joanne keys [42][x][10][=]420	Interviewer: Tell me why you’re choosing these numbers. Why did you just do five and now you’ve just done seven.
Joanne: Forty-two times ten, that’s too high so...	Nicki: Well, five there were fifty-two and that was too many, and so I tried seven.
Joanne keys [42][x][8][=]336	Interviewer: Why? What are the five and the seven about?
[pause]	Nicki: How many coaches.
Joanne: They’d need eight coaches, and they’d have...	Nicki: Eight now.
[pause]	Nicki keys [313][÷][8][=]39.125
Joanne: Twenty-three places left over.	Nicki: Eight and lots of seats left over.

Figure 3: Calculator-based trial-and-improvement strategies for the coach problem

From these examples it becomes clear that using a calculator is far from being the unthinking process of popular repute. It is a matter not simply of operating the machine, but of formulating computations and interpreting their results. In the case of division, this involves understanding the relationship of different forms of the operation to the particular form carried out by the machine. Moreover, when more complex sequences of computation are carried out, structuring these and recording their results may play an important part in effecting and interpreting the calculation with success. Consequently, a calculator-aware number curriculum needs to plan for the development of such expertise in calculator use; not assume that little expertise is involved, or that pupils will pick it up informally.

Indeed, a parallel can be drawn with the way in which long division is presented as a capstone of the traditional elementary number curriculum; not just as a crowning achievement in column arithmetic towards which pupils aim, but as a curricular organiser drawing on—and so having the potential to draw together—many important curricular strands. Both pedagogically and politically, one of the weaknesses of CAN may have been the absence of a corresponding calculator-based procedure to act as crowning achievement and curricular organiser. Elsewhere I have outlined how co-ordinated empty-number-lines provide a means of recording the whole range of informal strategies for quotient-and-remainder division—mental, calculator and hybrid

(Ruthven 2001). This evolving scheme has the potential to support a learning trajectory in which informal strategies are structured, curtailed, reorganised and refined, leading eventually to an efficient and systematic calculator-based method of quotient-and-remainder division.

Key Lessons for Policy and Practice

An important lesson to be drawn from the work reported here is the extent to which using a calculator successfully to tackle number problems is a mindful process. The user must formulate computations in terms appropriate to the calculator, and often monitor and moderate the steps of such computations. The user must then interpret the results provided by the machine, and sometimes translate them into a different mathematical or situational form. Consequently, a calculator-aware number curriculum must plan for the development of such expertise; not assume that little is involved, or that pupils will pick it up informally.

A broader lesson is that a 'calculator-aware' number curriculum is much more than a conventional number curriculum with calculator use 'bolted on' to it. Nor is it a wholly 'calculator-based' one. While calculators replace standard written methods as the computational means of secure resort, children's strategies of mental—and informal written—calculation retains significance as means by which number concepts can be actively developed. Equally, various forms of calculator use are employed to stimulate and support children's exploration of properties of number. Here again, such an approach requires careful planning, particularly of curriculum sequences to underpin continuity and progression in children's learning; it cannot simply be improvised around a conventional curriculum.

References

- Foxman, D. (1996). *A comparative review of research on calculator availability and use: Ages 5-14*. Unpublished report to School Curriculum and Assessment Authority.
- Plunkett, S. (1979). Decomposition and all that rot. *Mathematics in School*, 8(3), 2-5.
- Ruthven, K., Rousham, L. & Chaplin, D. (1997). The long-term influence of a 'calculator-aware' number curriculum on pupils' mathematical attainments and attitudes in the primary phase. *Research Papers in Education*, 12(3), 249-282.
- Ruthven, K. & Chaplin, D. (1997). The calculator as a cognitive tool: Upper-primary pupils tackling a realistic number problem. *International Journal of Computers for Mathematical Learning*, 2(2), 93-124.
- Ruthven, K. (1998). The use of mental, written and calculator strategies of numerical computation by upper-primary pupils within a 'calculator-aware' number curriculum. *British Educational Research Journal*, 24(1), 21-42.
- Ruthven, K. (2001). Towards a new numeracy: The English experience of a 'calculator-aware' number curriculum. In Anghileri, J. (Ed.), *Principles and practice in arithmetic teaching*. Buckingham: Open University Press.

- Shuard, H., Walsh, A., Goodwin, J. & Worcester, V. (1991). *Calculators, children and mathematics*. London: Simon & Schuster.
- van den Brink, J. (1993). Different aspects in designing mathematics education: Three examples from the Freudenthal Institute. *Educational Studies in Mathematics*, 24(1), 35-64.

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Part 3

The Teaching of Arithmetic

Introduction

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Part 3 considers the teaching of arithmetic in the light of recent research into children's thinking and the easy availability of electronic computation devices.

Stacey emphasises the need to rethink with care the place of the teaching of algorithms in schools. She places the need for algorithms in its historical context. She argues that the coming of electronic calculating devices provides the opportunity, indeed the responsibility, for society to decide which computational methods to teach in schools, and she provides some criteria for making such decisions. She draws attention to some potential dangers in withdrawing the teaching of all algorithms, and also reminds us that discussions about the role of algorithms should not be confined to considerations of whole number algorithms and the primary school.

Reys and Reys describe the role of estimation within the development of computational skills and number sense, and draw attention to the contrast between official recognition of its importance and the lack of structured approaches to its development. After an historical overview of estimation in school texts, they describe characteristics of an appropriate contemporary approach to estimation within the framework of developing numerical skills and number sense. They emphasise the importance of developing estimation skills and understandings alongside the teaching of (particularly mental) computation.

Trafton and Thiessen describe an approach to computation that emerged from collaboration between primary grade teachers and university faculty over many years. They show how in the project classrooms computational strategies and skills arose from treating computation as a problem-solving activity conducted over time, based on conceptual understanding and sense-making, in a classroom atmosphere that fostered reflection and communication. They describe some instances of how the traditional written computation algorithms were incorporated in the approach.

Gravemeijer, van Galen, Boswinkel and van den Heuvel-Panhuizen take the argument further by proposing that the conventional written algorithms should be replaced by the teaching of 'semi-informal' routines, grounded in well-developed number sense. They agree with Stacey's view that it would be unwise to remove the idea of algorithms entirely from the number curriculum. Instead, they have provided instances from classrooms of how more 'natural' written algorithms can follow on from children's informal mental computation.

Dole shows that the approach shared by all the authors in this book is not confined to whole numbers or to the primary school. She takes the difficult case of percents and, after discussing the weaknesses of traditional approaches, indicates how an approach based on conceptual understanding of percent can be allied with a diagrammatic visualisation to provide a practical and effective approach.

Rethinking the Algorithms of School Mathematics

Kaye Stacey

University of Melbourne

The thinking about teaching and learning arithmetic that underpins this book comes from weaving together three threads. The first thread comes from research into children's thinking. Studies of how children approach number problems lead us towards an arithmetic that capitalises on the strengths of human thinking and builds upon intuitive ideas. This mathematics may be easier to teach and to learn and hence achieve the goals of mathematics education more fully and more often for more children. The second thread arises from studies of children's performance on arithmetic tasks, in school and outside. It is clear that in many everyday tasks a broadly defined sense of number is invaluable. Sometimes this number sense supplements accurate calculation, as when there is on-line monitoring of whether an answer is about the right size. Number sense can also be used to replace accurate calculation in everyday contexts when only an approximate answer is needed. Developing number sense therefore needs to become a goal of school mathematics, sitting alongside goals of computational proficiency. The third thread is external. The advent of calculators that are sufficiently robust and inexpensive that even children can own them means that many constraints on the mathematics curriculum have been lifted. No longer is there a social demand for children to leave school adept at calculation for commercial purposes. There is a new freedom to rethink the mathematics that children are taught at school. The third thread provides us with an opportunity, while the first and second threads advise us on how to use the opportunity well.

This chapter takes a broad look at the changes that have come about and which still need to occur in response to this opportunity. It begins by discussing what an algorithm is, in a mathematical sense, pointing out that the range of algorithms taught at school is much wider than those for "the four processes" of whole number arithmetic. In many applications, efficiency is one of the most important features of an algorithm, but now in school mathematics, only some algorithms need to be highly efficient, and others need not be taught at all. However, deciding that an algorithm need not be taught at all can have unforeseen consequences and this is illustrated with the case of the fraction algorithms. Some of the purposes which fraction algorithms played in the curriculum have been overlooked and new ways of attending to them are needed. There is also a need for some algorithms to be updated to integrate calculator use.

What is an Algorithm?

Teachers generally use the word ‘algorithm’ to describe the written methods for adding, subtracting, multiplying and dividing whole numbers as traditionally taught in schools. Some teachers of younger children use the word to describe problems such as $9 + 5$ written vertically, as a precursor to the standard setting out of column addition. In this chapter, the mathematical meaning of ‘algorithm’ is adopted. The term encompasses *any completely specified procedure that produces a desired output for any specified class of inputs*. In this sense, column addition and subtraction, and long multiplication and division of whole numbers are all algorithms. Given any pair of whole numbers, the specified steps of the algorithm can be followed, finally giving the desired result. The input is the set of pairs of whole numbers (and not zero in the case of division), the rules are specified, and provided the steps are followed accurately, the output is the required answer. School arithmetic has also traditionally contained many other algorithms, such as multiplying by ten by adding zeros or moving the decimal point, the “invert and multiply” rule for dividing fractions, the rule using the “lowest common multiple” for the addition of fractions, solving simultaneous equations by substituting, and cross-multiplying for ratio problems.

Perhaps the first computational task confronting children is to find out how many objects there are when two groups are combined. The earliest procedure that children adopt is to count all. A child who knows there are 3 in one set and 5 in the other will put them together and count from 1 to 8. This is an algorithm: it is a completely specified procedure that produces a desired output for any two natural numbers. As the child grows older, he or she will learn to count-on from 3: counting 3 (for the first set), 4, 5, 6, 7, 8. Later the child will know to count-on from the larger; counting 5 (for the larger set), 6, 7, 8. These are three algorithms for finding the number of objects in the union of two sets and they are of increasing sophistication and efficiency. Programs such as Count Me In Too (Mulligan, *this volume*) place a high priority on moving children towards the most efficient of these methods. Later, because adding small numbers is required for so many other tasks, children move beyond these algorithms, learning to respond without counting.

The standard written algorithms taught in schools for the four operations have also been selected from a range of possible algorithms. Standard long multiplication, for example, relies upon knowing multiplication table facts from 0×0 to 9×9 and being able to add. However, the “Russian peasant” method requires only the ability to double and halve numbers and to add. No knowledge of multiplication facts is required. The two algorithms are contrasted in Figure 1. In the alternative algorithm, one number is halved (ignoring remainders) and the other is doubled, and the multiplication product is found by adding the numbers corresponding to odd numbers in the halving column.

Standard Algorithm	Russian Peasant Algorithm
$ \begin{array}{r} 86 \\ \times 38 \\ \hline 688 \\ 2580 \\ \hline 3268 \end{array} $	$ \begin{array}{rcl} 38 & & 86 \\ 19 \longrightarrow & 172 & \longrightarrow 172 \\ 9 \longrightarrow & 344 & \longrightarrow 344 \\ 4 & 688 & \longrightarrow 2752 \\ 2 & 1376 & \longrightarrow 3268 \leftarrow \text{ANSWER} \\ 1 \longrightarrow & 2752 & \end{array} $

Figure 1: Examples of two different algorithms for multiplication of 86 and 38

The Efficiency of Algorithms

Algorithms vary in efficiency. Finding prime numbers is a good example that is simple enough to discuss at school and yet still occupies professional mathematicians. To test whether any whole number (except 1) is prime, one algorithm is to check if it is exactly divisible by any of the whole numbers less than itself. If no number other than 1 divides it exactly, the number is prime. To find out that 127 is prime using this algorithm, divisions by all the 125 numbers from 2 to 126 are needed. This algorithm is very simple to remember, but it is very lengthy. A more efficient algorithm is to test if 127 is exactly divisible by any number less than its square root. To test if 127 is prime using this algorithm, it would only be necessary to carry out divisions by the ten numbers from 2 to 11 because 12×12 is bigger than 127. An even more efficient algorithm is to see if the number is exactly divisible by any of the prime numbers less than its square root. To find out if 127 is prime with this algorithm, it would only be necessary to carry out five divisions (by 2, 3, 5, 7 and 11). (Of course this algorithm is only more efficient if a list of all the smaller prime numbers is already at hand.) Today, there are many sophisticated algorithms for finding prime numbers, because large prime numbers are used for coding and decoding messages, with many applications such as securely sending telephone conversations, email, electronic banking, and national security.

Constructing very efficient algorithms to use in huge computer programs is a major task of working mathematicians. When billions of calculations have to be done repeatedly, as they are for everyday applications such as weather forecasting, transport scheduling or transmitting information, efficiency is critical. Efficiency is much less important in many other circumstances, especially given the power of modern technology. For example, I have just verified that $2^{2^5} + 1$ is not a prime number using a spreadsheet: $4294967297 = 641 \times 6700147$. The total time for opening the spreadsheet, programming it, making a few mistakes along the way and getting the result was seven minutes. This is interesting because, in just a few minutes, using the technology in my house, I repeated a major computational feat by one of the greatest mathematicians who has ever lived. About 1630, Fermat was looking for some way of predicting which numbers would be prime. He noticed that the numbers 3, 5, 17, 513 and 65 537 are all prime and that these numbers belong to a special pattern: $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 513$ and $2^{2^4} + 1 = 65537$.

From this, he conjectured that all the numbers of this type would be prime. This observation presented the possibility that there may be a pattern amongst the apparently random occurrences of prime numbers. Here, mathematicians were stuck for about 100 years until Euler managed to check the next case. He found that $2^{2^5} + 1$ was not a prime number, disproving Fermat's hypothesis. The smallest factor was 641. This example of prime numbers shows that the importance of efficiency of algorithms depends on the tools available.

Not all Human Computation is Algorithmic

It is important to recognise that people of all ages can do problems and computations in a variety of *ad hoc* methods, without using algorithms. For example, if a class of young children was asked to add 48 and 52, it is highly likely that at least one child will know the answer is 100 because he or she sees that 48 is 2 less than 50, 52 is two more than 50, and $50 + 50 = 100$. This demonstrates fine understanding of the important 'compensating' property of adding. However, relying on special coincidences like this is not using an algorithm. Algorithms are valuable precisely because they give a routine that works in a prescribed way for all elements of a specified domain. People can use them on any problem that fits the conditions and they can be programmed into machines. When deciding curriculum issues in the future, it will be important to weigh up the advantages of children learning an algorithm rather than being expected to use the many *ad hoc* computational methods that are available to a child with good number sense.

New Freedoms for Selecting School Algorithms

Before the age of cheap electronic computation, the selection of arithmetic algorithms to teach in schools was highly constrained and most especially so for the most-used operations (addition, subtraction, multiplication and division of whole numbers). Arithmetic algorithms needed to be highly efficient, because they would be carried out many times every day by people working in many businesses. Algorithms also needed to be written compactly to fit neatly into business ledgers. Carry or crutch figures and crossings out that would make the entries hard to review or audit were undesirable. For these reasons, until calculating machines were widely used in commerce, the subtraction algorithms of equal addition performed without visible crutch figures, was preferred over decomposition (see Figure 2). This was despite influential and well designed research by Brownell and Moser (1949) and others that children found decomposition easier to understand.

Equal Addition Algorithm	Decomposition Algorithm
$\begin{array}{r} 832 \\ - 184 \\ \hline 648 \end{array}$	$\begin{array}{r} 7 \quad 12 \\ \cancel{8} \quad \cancel{3} \quad 12 \\ - 184 \\ \hline 648 \end{array}$

Figure 2: Two different algorithms for subtraction.

The chapter from Gravemeijer, van Galen, Boswinkel and van den Heuvel-Panhuizen (*this volume*) shows how investigations into child-friendly methods can be taken much further. The classic written algorithms of column arithmetic for addition, subtraction (either equal additions or decomposition), multiplication and division are highly compressed and often taught rigidly even where minor variations are possible. As Bell, Costello and Kuchemann observed (1983), following rigid sets of rules is something at which machines excel but humans do not. The chapter by Gravemeijer and his colleagues demonstrates the potential to think broadly about a wide range of candidate algorithms, some of which will look very different to the ones we have taught in the past.

The advent of the calculator has also narrowed the range of algorithms that need to be taught. For example, finding the square root of a number is important for very many applications. These include simple common tasks such as making a square of given area or finding the length of the diagonal of a rectangle using Pythagoras's theorem. The by-hand algorithm for finding a square root is rather like long division, although more complicated (see Figure 3). This was taught until early in the twentieth century, when it was replaced by the use of logarithms. Later, in the brief period before four-function calculators had a square root button as a standard feature, iterative algorithms seemed the way of the future. The progression of these algorithms over just a few decades is illustrated in Figure 3. Today we teach no algorithm other than "press the square root button". It seems unlikely that people today understand square roots inadequately because they cannot calculate them by hand. A good understanding of the meaning of the square root and the main properties seems quite sufficient. A vast body of research, including the studies of the impact of calculators in the earliest grades (see Groves; Ruthven, *this volume*), has confirmed that providing children with calculating devices does not eliminate the need to acquire a deep understanding of the need for the computation, for how the results work, or to develop an intuitive sense of its properties.

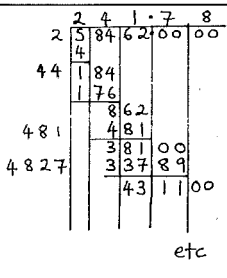
1935 By hand	1955 Using logarithms	1975 Calculator iteration	1995 By calculator
 <p>etc</p>	$\log_{10} 58462$ $= 4.767$ $\frac{1}{2} \log_{10} 58462$ $= 2.383$ $= \log_{10} (\sqrt{58462})$ Use antilogs: $\sqrt{58462} = 241.789$	$\frac{200}{2} (200 + \frac{58462}{200}) = 246.655$ $\frac{1}{2} (246.655 + \frac{58462}{246.655}) = 241.837$ $\frac{1}{2} (241.837 + \frac{58462}{241.837}) = 241.789$	$\sqrt{58462}$ 241.789

Figure 3: School algorithms for square roots evolved over the twentieth century

The Case of Fraction Algorithms

A straightforward strategy for increasing benefits and spreading costs of teaching any one computational method is to achieve several goals at once. Choosing a computational method that embodies a principle important elsewhere, for example, makes good sense. Alternatively, unnoticed disadvantages can arise when a change at one point of the curriculum takes away important groundwork for work on something else. Because the teaching of mathematics has evolved over many years, the various parts can depend on each other in ways that may not be appreciated. This is the main danger in making radical change in the curriculum—that we may not appreciate what students learn that is incidental to the explicit goal. Here the environment is a good analogy. Losing a species or introducing a new one can both have unforeseen effects on a functioning ecology. Careful environmental impact statements for the mathematics curriculum need to explore the consequences of major changes in the mathematics curriculum. These ideas will be illustrated below with the example of fraction algorithms.

In Australia, the introduction of decimal currency in 1966 and the subsequent change from imperial to metric units of measure changed the relative importance of fractions and decimals in the primary school curriculum, and incidentally saved a great deal of curriculum time (estimated at 18 months in six years, by the Australian Council for Educational Research (1964)). Decimals became more prominent than before in everyday calculation, and soon calculators provided an easy way to deal with them. Figure 4 demonstrates that by hand, finding two thirds of a year with a fraction method is markedly easier than with a decimal method. With a calculator, the two methods are quite comparable: thinking of two thirds as a decimal is as easy as dividing by 3 and multiplying by two.

By fractions, by hand	By fractions, by calculator	By decimals, by hand	By decimals, by calculator
$\frac{2}{3} \times 365$ $= \frac{730}{3}$ $= 243 \frac{1}{3}$	$\frac{2}{3} \times 365$ $2 \times 365 = 730$ $730 \div 3 = 243.33$	$\begin{array}{r} 365 \\ \times 0.667 \\ \hline 21900 \\ 219000 \\ \hline 243.457 \end{array}$	365×0.667 $= 243.457$

Figure 4: Finding two thirds of a year by fractions or decimals

This relative shift in the importance of decimals and fractions was quickly reflected in the primary school and junior secondary curriculum. For example, by 1995 in Victoria, students are to use everyday fractions (i.e., those with small or ‘round’ denominators); the algorithms for addition and subtraction are generally only practised on fractions with the closely related denominators; and multiplication and division are done without cancelling and delayed into secondary school—if they are taught at all (Board of Studies, 1995). The steadily reducing emphasis on fractions was often justified by assertions that operations on fractions are rarely required in everyday life. Emphasis in primary arithmetic shifted onto developing the concept of what a fraction is, and how it can be conceived as a number. Evidence that policies such as this have been adopted widely around Australia is obtained from the *Third International Study of Science and Mathematics*. Whilst Australian performances are generally above the international average, on fraction division the performance (25% correct) on the item $\frac{8}{35} \div \frac{4}{15}$ was well below the international average (Lokan, Ford, & Greenwood, 1996).

Which Baby was tossed out with the Bath Water?

It may well be the case that operations involving rational numbers are rarely used in everyday life, and that where they are used, decimals can very often be used instead. For example, a measured quantity, such as how much flour for a recipe, may be given as a fraction but can equally well be given as a decimal. Decimal calculations can easily replace fraction calculations to find the cost of two and a quarter kilograms of flour at ninety cents per kilogram or the area or perimeter of a rectangular piece of land. However, this analysis misses the use of fractions that pervades algebra and the arithmetic of secondary school. This is the use of fractions as a notation for division, ratios and proportions. The fraction algorithms are significant in this context because they embody how multiplication and division, addition and subtraction mix together. This point is illustrated with two examples.

The formula for the circumference of a circle given its radius is well known as $C = 2\pi r$. It is a simple task to transform it to obtain a formula for the radius: $r = \frac{C}{2\pi}$. Despite the notation, the quantity $\frac{C}{2\pi}$ is neither a ‘fraction’ nor a ‘rational number’ in any ordinary sense of the word. The denominator 2π , for example, is certainly not an integer, as it would have to be for a classic fraction. Instead, fraction notation is being used here as the normal algebraic notation for division. Secondary school mathematics

uses fraction notation for ratios or proportions and for quotients of any division; not just for 'fractions'. Importantly, any quantity of this nature—rational number or not—obeys the operation rules learned for fractions.

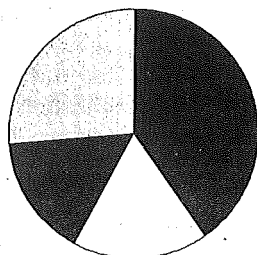
There is a lot to know about even a simple expression such as $r = C/2\pi$. For example, can you work it out (on a calculator or otherwise) by doing $C \div 2 \div \pi$ or $C \div 2$ multiplied by π or should you first multiply 2 by π and then divide C by this answer? Unfortunately, learning fraction algorithms probably prepared only a tiny minority of students to deal adequately with questions like these. However, simply removing most of the work with fraction algorithms from the curriculum has meant that there is almost no structure provided for students to think about these issues. We now need (but have not found an agreed way to provide) serious attention to important properties of the arithmetic operations that were embedded in the rules for operating on fractions. For example, the rule $a/b + c/d = (ad+bc)/bd$ can be viewed just as an algorithm for adding two fractions. On the other hand, it is also telling that addition and division must be mixed with care: to change from dividing first then adding (LHS) to adding first then dividing (RHS) special rules need to be followed.

Curriculum Change only Half Finished

This second example illustrates the rather unsatisfactory half-way house that we have now entered as arithmetic with a calculator is grafted onto former practices. One day, I sat with Penny, a Year 8 girl, drawing a pie chart to display data about the modes of transport to school of 57 students (see Figure 5). Penny's table showed that 23 of the students came by car and so, following the method in the textbook, to calculate the angle of the car's slice of pie she took out her calculator and entered 23, divided it by 57, multiplied the answer by 360 and divided the answer by 1, obtaining the correct answer of 145.26. Next, she treated the data on students who came by train similarly: 18 divided by 57, multiplied by 360 and divided by 1, obtaining the answer of 113.68. As Penny continued with this process, I suggested she calculate $360 \div 57$ and keep this in the memory to multiply successively by 23, 18, etc. She was interested in this suggestion but was not sure if it would give the right answers, so we talked about $360 \div 57$ giving the number of degrees on the pie chart for each student and that seemed to help.

Transport	Students
Car	23
Train	10
Bus	9
Walk	15
<i>Total</i>	<i>57</i>

Transport to school



■ Car 23 □ Train 10 ■ Bus 9 □ Walk 15

Figure 5: Penny draws a pie chart

My comment here is on the method suggested by the textbook. Like the alternative method which I explained, the textbook method is easy to explain: to find the size of the sector for car travel, take 23 fifty-sevenths of a whole circle (360 degrees) because 23 fifty-sevenths of the students come by car. However, the textbook authors have chosen a fraction method and setting out for what is no longer a fraction question. An efficient method with a calculator can look quite different, as I tried to explain to Penny. Calculate the number of degrees per student, put it in the memory and multiply in turn by each data entry. The textbook might write this as $[360 \div 57] \times 23$; a general case of $[360 \div \text{number of observations}] \times (\text{number in category})$.

For the calculator age, we need to adjust the methods that we teach, the way that we communicate them to students and the way that students are expected to record their work. Now, well over two decades since the advent of accessible calculators, we should begin to see calculator-designed methods up front in work like this with a sensible calculator-friendly notation.

Agenda for the Future

The school arithmetic curriculum should be routinely reviewed for the foreseeable future. This is a time of major re-adjustment and it is unlikely that the technology available even in primary schools is stable for the foreseeable future. We are living in a time where there needs to be regular assessment of the costs and benefits of all the algorithms taught at school (not just column arithmetic). As noted above, consideration needs to be given to the on-going importance of the primary skill being learned, in the light of the amount of effort that goes into teaching so that pupils have success. However, the secondary goals that are achieved or not achieved by teaching or omitting to teach any given computational method also need careful consideration. A major task is to identify both the major and secondary goals that the old algorithms served in the curriculum and to think about filling the gaps (if any) that they leave.

We live in a time when computation is still being revolutionised. For the moment, I am content that considering the triumvirate of mental methods, pencil and paper methods and four-function calculator methods serves us well for thinking about computation in elementary arithmetic. However, already more capable computational aids such as spreadsheets and computer graphing need to be considered for older students. It is also possible that widespread accessibility of smarter tools may impact yet again on primary arithmetic. Because of our place in history, decisions that we make now are likely to be unstable. However, the principles on which we make decisions will continue to be useful.

At this time, we still need creative work to design new computational methods to be taught in school and research to evaluate their effectiveness from a comprehensive perspective. As is clearly demonstrated throughout this book, research into child-friendly methods has begun and good progress can be expected from it. Similarly, this book shows that we are beginning to learn where a formal algorithmic procedure is valuable and where children can generally rely on ad hoc 'number sense' methods, which vary according to what numbers are involved and what relationships can be quickly spotted. On the other hand, thoroughly adapting the very large number of computational methods that we teach so that they are really suited to the computational tools that we expect students to use has only begun.

References

- Australian Council for Educational Research (ACER) (1964). *Primary school mathematics: Report of a conference of curriculum officers of state education departments*. Melbourne, 16-20 March, 1964. Melbourne: ACER.
- Bell, A.W., Costello, J. & Kuchemann, D. (1983). *A Review of mathematical education* [Part A: Research on learning and teaching]. England: NFER-Nelson.
- Board of Studies (1995). *Curriculum and standards framework: Mathematics*. Melbourne: Board of Studies.
- Brownell, W.A. & Moser, H.E. (1949). *Meaningful versus mechanical learning: A study in Grade II subtraction*. Durham, NC: Duke University Press.
- Lokan, J., Ford, P. & Greenwood, L. (1996). *Maths and science on the line: Australian junior secondary students' performance in the Third International Mathematics and Science Study*. Melbourne: Australian Council for Educational Research.

Estimation in the Mathematics Curriculum: A Progress Report

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Over the past 25 years there has been strong, consistent and broad-based affirmation of the importance of estimation as a process of thinking and as a necessary skill for mathematical literacy. National reports and curriculum frameworks in many countries including the United States, Australia, Japan, Sweden and the United Kingdom have emphasised the need for more attention to estimation (National Council of Teachers of Mathematics, 1980; 1989; 2000; National Commission on Excellence in Education, 1983; Australian Education Council, 1991; Cockcroft, 1982; Japanese Ministry of Education, 1989; Emanuelsson & Johnson, 1996).

In the United States estimation became a focus of attention when reviewers of data from the first National Assessment of Educational Progress (NAEP) (Carpenter, Coburn, Reys, & Wilson, 1976) concluded that, “developing skill in estimation should be a major objective of the school mathematics curriculum ... In fact, estimation is more important than precise calculation for many common uses of mathematics. Given the importance of estimation, it is perhaps one of the most neglected skills in the mathematics curriculum” (p. 296). In 1977 the National Council of Supervisors of Mathematics (NCSM) released a position paper detailing a list of fundamental ‘basic skills’ for mathematics that included problem solving, reasoning, and estimation. This brief but influential document emphasised that: “Students should learn to inspect all results and to check for reasonableness... Students should be able to carry out rapid approximate calculations ... and to decide when a particular result is precise enough for the purpose at hand” (p. 20).

In a paper summarising the emphasis on estimation in the school mathematics curriculum, Buchanan (1978) argued that, in spite of its recognised importance, “estimation continues to slip through the structural cracks of school mathematics” (p. 6). He pointed out that in order to develop facility with estimation, “there must be a long-range strategy for sustained use and application ... that extends over several years of mathematics instruction”.

Despite the lack of instructional emphasis on estimation in schools, researchers noted that some students do, in fact, exhibit an inclination to review computed results, to judge reasonableness of answers, and to make estimates when exact values are either unnecessary or impossible to obtain (Reys, Bestgen, Rybolt, & Wyatt, 1982). Unfortunately, many more students do not view mathematics as a sense-making activity and do not have fluency in developing estimates or judging the reasonableness of results (Sowder, 1992).

Why do some students fail to develop estimation fluency including the inclination to reflect on computed results? Is this a reflection of the curriculum that students experience? Is it a reflection of the goals of mathematics learning that teachers overtly or subtly convey to students? Is it the case that some students are simply not mature enough to reason or reflect on computed answers or that they do not see the value in this reflection? Must students first become proficient in computation before they understand and can estimate? These are important questions whose answers are still only hunches in the minds of many educators.

In this paper we review progress made in addressing the call for more attention to estimation, including research that has influenced changes in curricular attention to the topic. We review the nature and extent of attention to estimation in instructional materials used in the U.S. over the past 25 years. Finally, we propose an instructional framework for helping students come to better understand and apply estimation concepts and strategies.

Estimation as Portrayed in Curriculum Materials

The nature and extent of emphasis on estimation in popular American textbooks has changed over the past 25 years. We summarise here the portrayal of estimation in textbooks published in the United States within three distinct periods (1972-1983; 1984-1994; 1995-present). Textbooks in each period treat estimation differently, having been influenced by increased calls for attention to estimation as well as research.

1972-1983

Buchanan's review of the treatment of estimation in textbooks used in the 1970s confirms that attention to estimation during this period was minimal. The average number of lessons devoted to estimation in five popular elementary textbooks then used in the United States was three or less per year (Buchanan, 1978). These lessons typically concentrated on whole numbers and included a standard method for rounding whole numbers (e.g., if the next digit is five or more, round up). Students developed skill in rounding whole numbers of varying size to the nearest ten, hundred, thousand, and were expected to use this method to estimate sums, differences and products. This trend in the treatment of estimation continued through the early 1980s. The publication of several influential documents including the NCSM list of basic skills (1977) and a study highlighting strategies used by good estimators (Reys, Bestgen, Rybolt, & Wyatt, 1982) provided momentum and direction for increased attention and a change in focus for estimation as curriculum developers produced new textbooks in the early to mid-1980s.

1984-1994

Textbooks published during the period 1984-94 reflect a more substantial emphasis on estimation, both in terms of the amount of time devoted to the topic as well as the nature of the instructional emphasis. However, while these textbooks included more lessons on estimation, the instructional approach differed greatly among textbook series. Differences in the approach to presenting estimation can be characterised with regard to the following questions:

- How is estimation taught? (e.g., Are specific estimation strategies taught directly then practised or are strategies initiated by students then discussed collectively?)
- What estimation strategies are modelled and developed? (e.g., Are estimation strategies other than rounding introduced?)
- When is estimation introduced? (e.g., Is estimation taught as a precursor or following attention to exact computation?)
- What types of problems are included in estimation instruction? (e.g., Are students encouraged to estimate with whole numbers, decimals, and fractions? Are students asked to estimate when an exact calculation can be mentally obtained? Is there a context provided to help students gauge the value and exactness needed for their estimate?)
- What is the nature of the teacher support materials for developing estimation? (e.g., Do the materials make it clear why estimation is being emphasised and how it should be taught? Is an answer key provided for estimation exercises? If so, what form does it take?)

During the period 1984-94, research was reported that had a direct influence on the nature of the instructional approach to estimation. One example of this influence is the addition of instructional attention to strategies other than rounding. Prior to 1984, rounding was the only strategy associated with estimation within instructional materials. Since 1984, strategies such as front-end, compatible numbers, flexible rounding and clustering are introduced along with rounding. Specific lessons in these texts illustrate these strategies. In most cases, estimation lessons follow the development of written algorithms, although in a few series, estimation is taught just prior to the development of algorithms for exact computation. In these cases, students are encouraged to make estimates prior to doing exact computation as a way to judge the reasonableness of computed results. Unlike textbooks prior to 1984, estimation in these texts is not limited to whole numbers. In fact, these texts generally include lessons where students are encouraged to use 'benchmark' fractions such as $\frac{1}{2}$ in order to estimate the sum or difference of fraction computation items.

The predominant instructional approach for developing estimation in the textbooks produced and used during the period 1984-94 called for teachers to illustrate a strategy with a couple of examples and then ask students to estimate using the strategy just illustrated. In a few instances, students were encouraged to reflect on a situation, develop a strategy for estimating a particular problem then share the estimate and the strategy with other students and the teacher. In most cases, once an estimation strategy was introduced there was no discernible pattern regarding the opportunity to apply estimation in later lessons. In fact, there were very few situations beyond those devoted to introducing or teaching a particular estimation strategy where students were either encouraged or rewarded for estimating.

During this period, the quality of exercises provided to practise estimation varied greatly among the textbooks. While some exercises challenged students to estimate, students were frequently asked to estimate in computational situations such as $403 + 113$, 2×499 , or $9.3 - 0.7$ where an exact answer could be produced mentally by most students. Such experiences are likely to contribute to confusion among teachers and students about the value of estimation, as well as when exact answers or estimates are appropriate.

The call for an increased focus on estimation certainly resulted in more attention to the topic in textbook materials during this period. However, adding estimation as a new 'topic' in the curriculum (accompanied by new objectives, terminology, algorithms, practice sets, and assessment items) did little to convey to students (or teachers) the importance or richness of estimation. Like problem solving, if taught in isolation as a separate skill, students are not likely to assimilate estimation as part of a larger problem solving, reasoning process.

1995-Present

Some textbooks currently in use in the United States have simply carried forward the attention and instructional approach just described. Other textbooks such as a set of standards-based materials developed with the support of the National Science Foundation approach estimation in a new way. Rather than focusing directly on estimation strategies, they provide contexts where mathematical skills and processes, including estimation, emerge from the problem-centred environment. That is, as students explore and solve a variety of problems, many within a real-world context, they have opportunities to use and discuss a range of estimation strategies.

A review of standards-based elementary and middle grade mathematics curricula (curriculum materials developed with support from the National Science Foundation and published after 1995) highlights a wealth of opportunities to estimate within different contexts. However, these opportunities are often subtle and may be overlooked or not fully developed by teachers. Furthermore, the ways in which teachers might exploit these estimation opportunities to promote conversations about estimation strategies are rarely developed in student and/or teacher resource books. While the problem-centred context for developing estimation appears to be a viable instructional model, more attention to helping teachers highlight estimation techniques and encourage students to discuss and share different estimation approaches is needed.

Student Performance on Estimation

Although instructional emphasis on estimation has increased over the past 25 years, estimation performance remains very low. This low performance has been documented by several national assessments in the United States (Carpenter, Coburn, Reys, & Wilson, 1976; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Dossey, Mullis, Lindquist, & Chambers, 1988). Several international research studies have also focused on student ability to estimate in computational settings. The results are surprisingly consistent (Reys, Reys, Nohda, Ishida, Yoshikawa, & Shimizu, 1991; Reys, Reys, & Flores, 1991; Reys & Yang, 1998). Estimation is generally not well understood or respected by students in countries represented in these studies. For some, estimation is synonymous with 'rounding numbers' according to a set of rigid rules. For others, it means guessing, but not necessarily an educated guess reflecting analysis and mathematical thinking. In Asian countries, exactness of answers is stressed and rewarded within mathematics instruction. This attention to exactness is in direct conflict with the nature of estimation and often results in confusion by students regarding the value and purpose of estimation (Reys, et al., 1991).

While estimation is generally considered an easier task than exact computation, research has consistently reported higher performance on exact paper/pencil

computation than on parallel items requiring estimation (Sowder, 1992). This phenomenon was reported in Taiwan, where over 60% of a sample of eighth-grade students correctly calculated the exact answer to $12/13 + 7/8$, yet only 38% of the same students made an acceptable estimate of '2' on a comparable estimation item (Reys & Yang, 1998). It is clear that those who champion the goal of high computational performance should look beyond performance on exact computation if they are interested in students' fluency with numbers and operations.

Research provides a context for understanding this phenomenon and helping students develop a mindset for estimation (Sowder, 1992; Markovitz & Sowder, 1994; Reys, 1984, 1986). Specifically, evidence suggests that many students have misconceptions about the value and purpose of estimation; that they have developed few, if any, strategies to estimate beyond the one strategy (rounding) taught in school; that they are typically more proficient at calculating exact answers than at estimating; and that some students do develop and utilise a variety of estimation strategies based not on instruction but their own reasoning and number sense. A brief elaboration of each of these findings follows.

Students (and oftentimes teachers) are confused about *what* estimation means. Students associate estimation with guessing an answer. For example, students asked to, "estimate the number of pennies in a jar" might equate this task with, "guessing the number of pennies in a jar". The notion of estimation as guessing then gets applied to "estimating the sum of two numbers" and translated to "guessing the sum of two numbers". While there is logic to these connections, guessing and estimating are not synonymous, and care needs to be taken to ensure that students and teachers are sensitive to the language and substantive difference inherent in this language. Estimation is based on understanding the numbers and operations involved and often draws on specific strategies that are based on this understanding to produce an 'educated' guess.

Students (and oftentimes teachers) are confused about *how* estimation is done. Historically students have been presented exercises that asked them to "Estimate then calculate". Many students reversed these tasks to calculate and then round their answer to get an estimate. Consider the following conversation with a fifth-grader:

- Interviewer: Please make an estimate to $424 - 195$.
- Kara: Let's see, (after a pause) my answer is 229 so my estimate is 200.
- Interviewer: Please make an estimate to: $5462 \div 26$
- Kara: (After a pause) May I write it down?
- Interviewer: I would like for you to do it in your head.
- Kara: I can't find the answer, so I can't make an estimate.

While this approach (calculating an exact answer as a means for developing an estimate) resulted in a reasonable estimate (200), it suggests that Kara believes that an exact answer is needed in order to produce an estimate. This approach to estimation is popular, although very inefficient, and without intervention becomes the strategy of choice as it produces 'correct' responses. It also leads students to believe that estimation is not a reflective process but an algorithmic procedure of rounding exact answers. Furthermore, it leaves many students wondering about the worth of estimation.

Estimation strategies used by good estimators have been identified and characterised (Reys, et al., 1982). Among the powerful and effective strategies used by

students who have developed their own strategies based on number sense are: front-end, flexible rounding, compatible numbers, and use of benchmarks. In addition to strategies, a number of cognitive processes, such as compensation and reformulation, are an integral part of producing estimates. However, simply teaching students a set of new estimation strategies does not insure that they will adopt and utilise them in appropriate contexts. If strategies are the focus rather than situations that allow ideas and strategies to emerge and be discussed, it is unlikely these strategies will be assimilated into the students' long-term repertoire.

Charting a Course

Detailing an instructional sequence for a topic as amorphous as estimation is nearly as challenging as describing how problem solving should be developed. In fact, estimation has many parallels with problem solving. Schroeder and Lester (1989) characterise several approaches to teaching problem solving—teaching about problem solving, teaching for problem solving, and teaching via problem solving. As with problem solving, instruction related to estimation has often been directed at teaching students 'about' estimation. That is, teaching them strategies that experts use as they estimate. A more appropriate and productive approach to teaching estimation may be to teach 'for' estimation. That is, provide students with many opportunities to apply estimation in meaningful contexts.

Providing opportunities to estimate alone will not be enough. Students also need help in developing a variety of strategies. The development of strategies should emanate from the need to estimate rather than from a contrived situation where the purpose of estimation is unclear. Like problem solving, estimation involves an attitude or inclination to estimate as well as a set of skills and strategies. Estimation is a mathematical theme that should run across many topics of study and not be limited to a unit of study. When an algorithmic approach to estimation prevails, the effort may even be counterproductive—leaving students with a general dislike and distrust for the estimation process. To be truly effective, a careful integration of estimation into appropriate areas of curriculum (e.g., number, percent, geometry, probability, etc.) must occur.

Important Characteristics of an Estimation Program

The ability to estimate and to judge the reasonableness of computed results is based on understanding numbers—their relative size and various representations—and on understanding operations—what they mean and how they can be translated. In addition, students must develop an awareness of the role and nature of estimation as an important mathematical process. Attention to measurement estimation can begin in the early primary grades. Work on computational estimation should wait until students have acquired some specific number and operation concepts (e.g., place value, models for operations).

Figure 1 highlights a general outline of instructional emphasis related to estimation across the K-8 mathematics curriculum. Some particular components of estimation are described in more detail in the following sections.

Grade Band	Emphasis	Example(s)
Primary (K-2)	Mental computation	$10 + 20$; $5 + 5 + 15$; $23 + 42$
	Estimation of quantity	Do I have more or less than 30 pencils?
	Measurement estimation	About how many centimeters long is this pencil?
Intermediate (3-5)	Mental computation	$5 \times 25 = (5 \times 20) + (5 \times 5)$, or = 5 quarters, or = $(5 \times 30) - (5 \times 5)$
	Knowing when an estimate is appropriate	In which of these situations is an estimate appropriate?
	Estimation strategies	Front-end, rounding to compatible numbers
	Benchmarks	Are these fractions smaller or larger than $\frac{1}{2}$?
Middle grades (6-8)	Estimation strategies	Compatible numbers, clustering
	Apply estimation to other mathematical topics	Angle measure, percent

Figure 1: Suggested instructional sequence related to estimation

Encourage the Development of a Variety of Estimation Strategies

Students should develop a variety of estimation strategies so they can select a strategy based on the context of the problem and the numbers and operation/s involved. Instructional programs should encourage the development of strategies such as front-end, compatible rounding, clustering, and use of benchmarks. Estimation strategies may evolve from discussions as students develop estimates and share their thinking. Strategies may also evolve from instruction initiated by the teacher related to specific strategies. The sequence in which strategies emerge from the work of students or are presented by teachers can be guided by instructional materials rich in opportunities and mathematics contexts that lend themselves to estimation. By the end of grade 8 students should have a range of estimation strategies they can call upon to estimate with whole numbers, fractions, decimals and percents. Students should have opportunities to apply, refine and extend these strategies throughout the secondary school mathematics program. Consider for example the problem shown in Figure 2. This example illustrates

how the same data were treated differently to produce slightly different estimates. Often different estimation strategies applied to the same problem produce different values, each of which is an acceptable estimate.

There is a 23% tax (federal, state and local) on gasoline. About how much tax is paid when it costs \$32.59 to fill the tank?

Rustin: \$32.59 is about \$30 and 20% of \$30 is $\frac{1}{5}$ of 30 or \$6.00.

Whitney: 23% is about 25%, so it is about $\frac{1}{4}$ of \$32 or \$8.

Nicole: \$32.59 is between \$30 and \$35, and 23% is about $\frac{1}{5}$ of \$30 or \$6, one-fifth of \$35 is \$7, so between \$6 or \$7.

Figure 2: Different approaches to the same estimation task

Establish a Mindset Regarding Estimation

Estimation by its very nature implies a degree of imprecision as illustrated by the different yet acceptable estimates in Figure 2. One of the challenges of teaching estimation is ‘reconditioning’ minds which have been directed to value one correct answer in mathematics. This means that care must be taken to avoid the ‘one right answer’ syndrome. The question, “Who has the right estimate?” must be viewed as irrelevant and replaced by, “Is your estimate reasonable?” Asking students to identify acceptable or reasonable intervals for an estimate is an important activity. Discussing intervals allows students to gain insight into other solution strategies and become more comfortable with the notion that several different estimates, each reasonable, may exist for a given situation.

Focus on when an Estimate is Appropriate

Experiences need to be provided to help students differentiate between situations requiring exact values and those where an estimate is sufficient. An example is given in Figure 3.

Binder \$8.76	Pen \$1.19	Carton \$0.89
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Kyle: I have only \$10. Can I buy these three items?

Scott: If I buy all of these items with a \$20 bill, how much change should I get?

Which situation calls for an exact answer? Why?

Which situation calls for an estimate? Why?

Figure 3: Examining contexts for estimation and exact answers

Some situations call for either an over-estimate or an under-estimate. Consider for example the situations shown in Figure 4. Deciding whether a situation calls for an over- or under-estimate is dependent on the particular context of the problem, so engaging students in considering a variety of contexts will facilitate their thinking and experience with a range of real world applications of estimation.

Rick: I am 285 km from the airport and the speed limit is 120 km/hr. About how much time should I allow?

Kelly: We are competing for a bid to remove snow from a driveway. Does it matter whether I estimate high or low for the job?

Does either of these situations call for an over-estimate? An under-estimate?

Why?

Figure 4: Examining contexts for over- or under-estimating

Emphasise Sense-making Adjustments to Initial Estimates

The process of adjusting a result is an important part of the estimation process. Adjustment or compensation is a natural process that complements all strategies and provides a means of refining initial estimates. For example, suppose an estimate for the area of a 27×38 room is needed. If students round the dimensions (27×38 to 30×40) they should realise that both values have been rounded up, so the resulting product is an over-estimate, as is shown in Figure 5. Making an adjustment of 1200 to yield an estimate of less than 1200, say 1100 reflects good number sense. Of course, another student may round the factors differently (e.g., 27 to 25 and 38 to 40) to produce an estimate of 1000. In this case, one factor was rounded up and the other down so it is not as easy to tell if the estimate is too high or too low, and the adjustment may not be necessary.

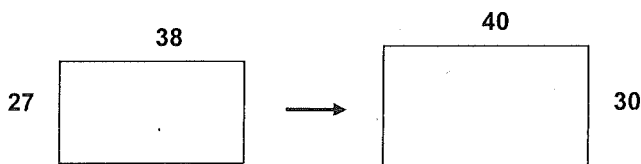


Figure 5: A model for consequences of rounding

Include Attention to Establishing Benchmarks

Benchmarks are common reference points that help estimators develop and judge initial estimates. For example, the student in the previous example who rounded 27 to 25 did so because 25 was close to 27 and is an easy factor to use in mental computation. It has properties such as being a factor of 100, which allow for various computational techniques. Use of benchmarks is critical for work with measurement as well as computational estimation. For example, a right angle is a common and very useful benchmark for judging the size of other angles. Good estimators routinely use benchmarks to judge the size of fractions. For example, recognising when a fraction is greater or less than a specific benchmark (e.g., one-half) is a critical part of conceptual development of fractions. Which of the following fractions are near 0? Near $\frac{1}{2}$? Near 1?

$$\frac{1}{9} \quad \frac{5}{6} \quad \frac{3}{7} \quad \frac{7}{15} \quad \frac{9}{10} \quad \frac{2}{5} \quad \frac{2}{19} \quad \frac{8}{9} \quad \frac{7}{8} \quad \frac{12}{13}$$

Thinking about each of these fractions in relation to fractions such as 0, $\frac{1}{2}$ and 1 encourages students to think about the relative size of these fractions. Such development provides important readiness for students to begin to operate with fractions. For example, students can reason that $\frac{7}{8} + \frac{12}{13}$ is almost 2, and that $\frac{5}{8} + \frac{8}{15}$ is a little more than one before they are asked to compute the exact answer. The payoff to this sensitivity is immediate, as students will have a notion of a reasonable answer before they are asked to apply algorithms, some of which are both tedious and complex. Reflection on the size of fractions provides a tool for alerting students to unreasonable answers. Decimals and percents use some of the same benchmarks as fractions. In addition, the benchmarks of 1%, 10% and 100% are important and useful in not only estimating but also in developing fundamental understanding of the concept of percent.

Summary

The sustained and strong attention to estimation over the past 20 years is recognition that students in the new millennium will have an increased need for this process. As use of technology such as calculators and spreadsheets continues to proliferate at home, in school and in the workplace, the need to estimate and recognise the reasonableness of results will continue to grow. If the mathematics curriculum is to reflect these needs, then significant improvement in how we help students develop estimation is needed. Our proposal for a mathematics curriculum that reflects greater attention to mental computation and estimation alongside the development of efficient methods to calculate exact answers is offered, so that students develop a range of efficient procedures for computing and that they also understand that different computational procedures exist and are valuable. As students develop ability to compute, they should simultaneously develop a deeper understanding of number and operations that will contribute to their overall mathematical literacy.

References

Australian Education Council. (1991). *A national statement on mathematics for Australian schools*. A joint project of the States, Territories and the Commonwealth of Australia, Australian Education Council and the Curriculum Corporation. Melbourne: Australian Education Council.

- Buchanan, A.O. (1978). *Estimation as an essential mathematical skill.. Professional paper 39*. Los Angeles: Southwest Regional Laboratory for Educational Research and Development.
- Carpenter, T.P., Coburn, T.G., Reys, R.E., & Wilson, J.W. (1976). Notes from the National Assessment: Estimation. *Arithmetic Teacher*, 23(4), 296-302.
- Carpenter, T.P., Corbitt, M.K., Kepner, H.S., Lindquist, M.M. & Reys, R.E. (1981). *Results from the second mathematics assessment of the National Assessment of Educational Progress*, Reston, VA: National Council of Teachers of Mathematics.
- Cockcroft, W.H. (1982). *Mathematics counts*. London: Her Majesty's Stationery Office.
- Dossey, J.A., Mullis, I.V.S., Lindquist, M.M., & Chambers, D.L. (1988). *The mathematics report card: Are we measuring up?* 17-M-01. Princeton, NJ: Educational Testing Services.
- Emanuelsson, G. & Johansson, B. (1996). *Kommentarer till kursplan i matematik, Lpo 94*. (Non-statutory guidance to the new Swedish curriculum of mathematics).
- Japanese Ministry of Education. (1989). *Curriculum of mathematics for the elementary school*. Tokyo: Printing Bureau.
- Markovits, Z. & Sowder, J. (1994). Developing number sense: An intervention study in Grade 7. *Journal for Research in Mathematics Education*, 25, 4-29.
- National Commission on Excellence in Education. (1983). *A Nation at risk: The imperative for educational reform*. Washington, DC: U.S. Government Printing Office.
- National Council of Supervisors of Mathematics. (1977). Position paper on basic mathematical skills, *Arithmetic Teacher*, 25(1), 18-22.
- National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston, VA: National Council of Teachers of Mathematics.
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- National Council of teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Reys, R.E. (1984). Mental computation and estimation: Past, present and future. *Elementary School Journal*, 84, 547-557.
- Reys, R.E. (1986). Evaluating computational estimation. In H. Schoen (Ed.). *Mental computation and estimation* (pp. 225-238). Reston, VA: National Council of Teachers of Mathematics.
- Reys, R.E., Bestgen, B.J., Rybolt, J.F. & Wyatt, J.W. (1982). Processes used by good estimators, *Journal for Research in Mathematics Education*, 13, 183-201.
- Reys, R.E., Reys, B.J., & Flores, A. (1991). Computational Estimation Strategies Used by Eighth Grade Mexican Students, *Educational Studies in Mathematics*, 22, 353-376.

- Reys, R. E., Reys, B.J., Nohda, N., Ishida, J., Yoshikawa, S., & Shimizu, K. (1991). Computational estimation performance strategies used by fifth and eighth grade Japanese students, *Journal for Research in Mathematics Education*, 22, 39-58.
- Reys, R.E. & Yang, D.C. (1998). Relationship between computational performance and number sense among sixth and eighth grade students in Taiwan, *Journal for Research in Mathematics Education*, 29(2), 225-237.
- Shroeder, T.L. & Lester, F.K. (1989). Developing understanding in mathematics via problem solving. In P.R. Trafton & A.P. Schulte (Eds.). *Elementary school mathematics* (pp. 31-42). Reston, VA: National Council of Teachers of Mathematics.
- Sowder, J. (1992). Estimation and number sense. In D.A. Grouws (Ed.). *Handbook of research on mathematics teaching and learning* (pp. 371-389). New York: Macmillan.

Linking Mental and Written Computation via Extended Work with Invented Strategies

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What should be the role of computation in contemporary math programs?

How should computation be taught?

What is the balance between written and mental computation?

What is the value of having students invent their own procedures?

Introduction

Rethinking computation is central in mathematics curriculum reform—its role, what it should include, and how it is performed. It is essential, both to redefine the role of computation so that it does not dominate school mathematics and also to find more appropriate ways to teach it.

However, the diversity of viewpoints about computation makes efforts to shift to better-balanced programs complex and contentious. Some argue that computation can only be learned through demonstration and practice with little attention to understanding. The opposite view is that children will naturally acquire skills through a focus on problem solving and understanding. Most reform documents argue computation is still important, but should be learned with understanding (NCTM, 2000). Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, and Human (1997), in a discussion of the implications of research on understanding and skill, support this view:

Learning computational skills and developing conceptual understanding are frequently seen as competing objectives. If you emphasize understanding, then skills suffer. If you focus on developing skills, then understanding suffers. We believe that this analysis is wrong. It is not necessary to sacrifice skills for understanding, nor understandings for skills. In fact, they should develop together. (p. 6)

Although there is evidence that understanding and skill support each other, it is not clear to teachers what the computational expectations are, nor how they can achieve a balanced approach. The need for clear guidelines is captured by Reys and Reys (1998):

Elementary teachers receive conflicting messages about the value of various computational techniques, mental and written, as well as about what strategies, invented and standard, should be introduced and developed at different levels... Often the messages related to mathematics instruction in general, and computation in particular are in conflict, and teachers are left to translate the mixed messages to their classroom practice. (p. 236)

Without guidelines, many teachers offer a dualistic program with one aspect emphasising understanding and problem solving and the other focusing on traditional computation (see Figure 1a). We need to bring these aspects together, so that teaching and learning is viewed through a single lens (see Figure 1b).

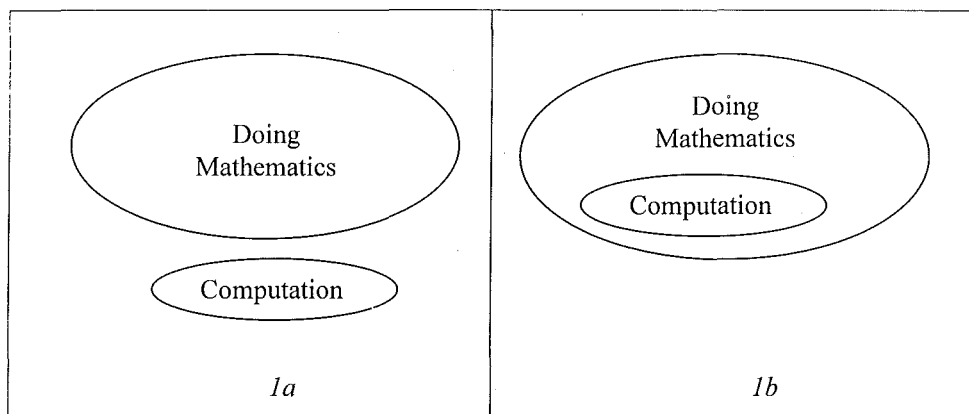


Figure 1: Relationship between doing mathematics and computation

This chapter presents an approach to computation that emerged from collaboration between primary grades teachers and university faculty over several years. In this work, computation develops within a problem-centred approach in which sense-making and children's thinking are emphasised.

The next section presents an overview of computation in classrooms that emphasise number sense. This is followed by an extended discussion of teaching computation through an ongoing emphasis on students' invented strategies. The final section discusses how standard algorithms can emerge from invented strategies.

Computation in Sense-making Classrooms

Our experience suggests that computation can be learned very differently than from under traditional approaches. Three key aspects of classrooms in which this occurs are discussed below.

Mathematical Sense-making

When making sense of mathematics is the primary goal, the classroom dynamics and work of teachers and students changes greatly (see a later section for a further discussion). First, learning often occurs through engagement in mathematically

significant problems and tasks. Children take ownership of tasks and work on them individually or in small groups using their own strategies. Teachers observe children to learn about their thinking and plan for class discussions.

Second, teachers conduct class discussions or seminars in which children share their thinking. This helps each child become aware of the rich variety of strategies that have been used. Teachers use seminars to teach by highlighting key ideas and conducting mini-lessons. These discussions have a central role in helping children develop understanding and strategies.

Third, teachers work to create a classroom community that supports reflection and communication. A learning community supports and values the thinking of all children, promotes mathematical risk-taking, and accepts mistakes as a natural aspect of learning.

Computation as Number Sense

A common view is that computation is applying well-defined rules for manipulating symbols. A contemporary perspective is that it is a dimension of number sense. Murray, Olivier, and Human note that in their curriculum project, computation is viewed “as a vehicle that students can use to increase their understanding of number and the properties of number and operations” (in Hiebert et al., 1997, p. 115). Individuals who possess number sense are able to navigate smoothly in the domain of number, using concepts and relationships in sensible ways to help them solve problems, interpret situations, and make judgments.

Number of the Day: 25	
Partitioning by Tens and Ones	Addition Pattern
10 + 10 + 5	20 + 5
	19 + 6
'Cancel Out' Strategy	18 + 5
25 + 5 – 5	15 + 10
100 - 100 + 25	
20 + 10 – 5	Subtraction Pattern
25 + 25 + 25 - 50	30 – 5
	31 – 6
Substitution	32 – 7
5 + 5 + 5 + 5 + 5	33 – 8
5 + 5 + 5 + 5 + (1 + 4)	35 – 10
(4+1) + (4+1) + (4+1) + (4+1) + (4+1)	100 – 75
	200 – 175
	500 – 475

Note: Children’s suggestions have been rearranged to highlight key strategies used.

Figure 2: Children’s ‘Number of the day’ suggestions

An example of the connection between number sense and computation is seen in the suggestions offered by children for 25, the number of the day (see Figure 2). An examination of their work shows that their responses primarily are based on number patterns and relationships, rather than the application of computational rules. In the

second example (see Figure 3) Nick used his number sense to solve $\$2.79 - 0.85$. Since he did not have a computational rule, he relied on his number sense to create a thoughtful strategy.

Handwritten work showing Nick's method to solve $\$2.79 - 0.85$. The work includes the equation $2.79 - 85 = 1.94$, a note $79 + 6 = 85$, and a detailed explanation of the strategy: $2.79 - 79 \rightarrow 2.00$, then 85 from 2.00 and it equals 1.15 , and finally $1.15 + 0.85 = 1.94$.

Figure 3: Nick's method to solve $\$2.79 - 0.85$

Learning Computation in Sense-making Classrooms

Computation unfolds very differently in sense-making classrooms. In traditional classrooms, it is taught in isolated units, instruction is teacher-directed and rule-driven, extensive practice is used to assure skill, and little attention is devoted to student strategies.

In sense-making classrooms, computation emerges in several contexts—a classroom event, a problem posed by the teacher, a problem created by a student, as well as through lessons. In one classroom, for example, the fourth graders figured out how 400 rulers that had been donated could be distributed equally among 12 teachers. In another room, children wrote word problems in the spirit of the book *Math Curse* (Szieszka, 1995) and solved each other's problems.

Solving problems and discussing computational strategies occur regularly. Children encounter problems involving all operations, and problems include both single and multiple steps. At times, problems focus on a particular operation. Children also come to view 'naked computation', (e.g., 12×24) as a problem, and treat these as thoughtfully as contextual problems.

We have attempted to capture the flavour of how computation can be learned in sense-making classrooms. It is important to emphasise that computation is not treated casually, as it remains an important goal. As a result, teachers give careful attention to children's growth in this area.

Building Computation on Students' Invented Strategies

This section describes how computational strategies can be built on students' ideas. A brief vignette that captures students' strategies for one problem sets the stage.

It was late October and the second grade children were about to participate in the school's mock election. One student asked how many second graders would be voting. The teacher turned this into a problem. She stated that there were 28 students in each of the two classrooms and challenged students to find the total. They quickly set to work. Some solutions were oral, some involved a written record, and some used tools such as hundreds charts and base-ten blocks. Figure 4 reveals the mathematically rich strategies

Oral Strategies

Ben: Well...I started with 28 and counted by tens because it's easy for me to use tens. 28 and 10 more is 38 and 10 more is 48. Then, 8 more is 56.

Marissa: 2 and 2 is 4, so 20 and 20 is 40. 8 and 2 of the other 8 is 10. Then 50 and 6 more is 56.

Laurie: I took 2 from one of the 28s and put it with the other 28. That made 30. I still need to add 26. 30 and 26 is 56.

Cody: 8 and 8 is 16. I added the 1 with the 2 and 2, so it's 56.

Elise: I thought of it as money. 28¢ is a quarter and 3 pennies. Two quarters is 50¢. 6 pennies more is 56¢.

Juan: I started with 25 + 25 because I just know that. 25 + 25 is 50. 3 and 3 is 6. 50 and 6 is 56.

Written Approaches

Caitlin: 20

20

40

16 (8 + 8 = 16)

56

Damarius: 20 + 20 = 40

8 + 8 = 16

40 + 16 = 56

Greg:

```
graph TD
    A[20] --- B[40]
    C[20] --- B
    D[8] --- E[16]
    F[8] --- E
    B --- G[56]
    E --- G
```

Allison: 30 + 30 = 60

60 - 4 = 56

Figure 4: Computational strategies for $28 + 28$

that emerged, both familiar and new ones (Trafton & Thiessen, 1999, pp. 52-53). Some strategies used children's knowledge of tens and ones, while others used patterns and relationships. A concluding seminar highlighted their thinking. Six key characteristics of this approach and how they promote student thinking and learning are now discussed.

1. The work is ongoing

Mathematics programs are typically organised by units dealing with specific content and skills. We have found it to be more productive to develop computational strategies throughout the year. Classrooms events, daily problem solving, and tasks or problems presented by teachers cause students to have ongoing experience with computation. This provides an extended *window of opportunity* for learning for *all* students; that is, they have the necessary time and experiences to become confident and skillful. Some students develop mature strategies quickly, while others progress more gradually. Over time, students grow closer together in their thinking and strategies.

2. Strategies come from the students

We often assume that direct lessons are necessary for students to learn specific strategies. However, strategies are part of the informal and intuitive mathematics that students acquire through their daily experiences. This 'street math' knowledge (Resnick, 1995) also reflects an understanding of important mathematical ideas and relationships. It is important to note that students' strategies make sense to them, often growing out of a problem's context and the specific numbers. The strategies also result from the emphasis on number sense. In our experience, most strategies that teachers want students to use are suggested by the students. However, if students don't suggest a strategy, teachers will offer it as another way of thinking and have children consider it.

3. Students learn strategies by using them in context and engaging in conversations about why they work

As we have noted, students learn strategies through repeated class discussions. Over time, then, *every* student has multiple opportunities to understand and use the thinking of *some* students. To highlight important strategies, teachers may build and post a list of key ones. They often have students examine the list to see which strategies were used and periodically add new ones to it.

Two factors, then, promote the use of productive strategies by *all* students. First, students regularly hear, discuss, and use them. Second, students are learning in classrooms in which reflection and communication are the norm. While each student may not understand every strategy, they use the ones that make sense to them and keep adding new ones to their repertoire.

Teachers note that all students make substantial growth over time in their thinking. The least mature students appear to pass through several stages as they shift from less mature to more mature approaches (see Figure 5). However, change occurs on the student's time schedule, not on the teacher's or textbook's schedule. Although some children make transitions slowly, teachers note that when mature approaches are internalised, a period of acceleration in a student's mathematical growth follows.

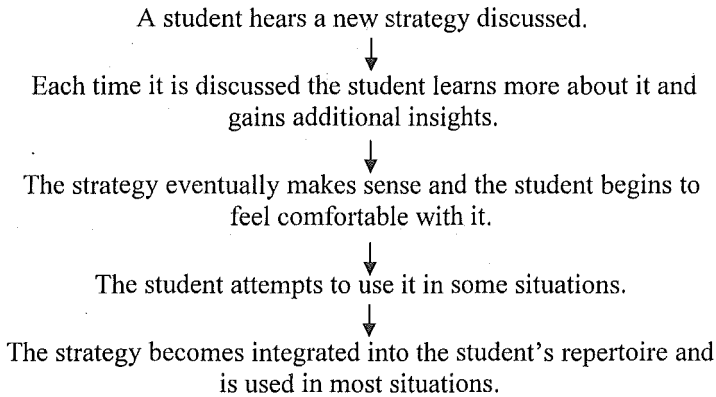


Figure 5: Learning path for strategies

4. Written records are emphasised

It is important for students to make written records of their thinking, in addition to sharing it orally. One reason is the benefit it provides for the student. Writing about one's thinking helps clarify and highlight the key elements of it. Resnick, Bill, Lesgold, and Leer (1991, p. 37) also note that "by using a standard mathematical notation to record conversations carried out in ordinary language and rooted in well-understood problem situations, the formalisms take on a meaning directly linked to the children's mathematical intuitions". Written notation also helps learners in the process of working out the steps of a solution (see Figure 6). Written records also promote communication with other children and adults. It helps everyone to make sense of another person's thinking and focus on the mathematics. Having a written record allows the student to review his/her work and enables the teacher to understand the person's thinking. It also helps in assessing progress over time.

6a: $\$2.79 - \$0.85 = \underline{\hspace{1cm}}$	6b: $\$1.35 + \underline{\hspace{1cm}} = \2.00
<p style="text-align: center;"> $\begin{array}{r} 279 - 85 \\ \hline 199 \\ \hline 194 \end{array}$ </p> <p style="text-align: center;"> $279 - 80 \rightarrow 199 - 5 \rightarrow 194$ </p>	<p style="text-align: center;"> $\begin{array}{r} 130 + 70 = 200 \\ 70 - 5 = 65 \\ \text{Check: } 135 + 65 = 200 \\ \hline 90 + 10 = 100 \end{array}$ </p>

Figure 6: Children's written records

5. Important mathematics is explored and learned

An examination of children's strategies shows application of important mathematical ideas, as illustrated in Figure 7. In Figure 7a one student used her knowledge of grouping and place value to partition 142; in 7b the student understood partitive division and then subtracted multiples of ten and one hundred. In 7c the student translated 12×40 to 6×80 by doubling one factor and halving the other. In finding the product the student showed that he understood that 6×80 is equivalent to doubling three eighties. Thus, far more is learned than how to compute. Significant mathematics emerges from the student-created algorithms, illustrating that computation can be a setting for exploring mathematics.

7a: $200 - 142$	7b: $429 \div 3$	7c: 12×40
$200 - 100 = 100$	100 100 100	12×40
$100 - 40 = 60$	20 20 20	6×80
$60 - 2 = 58$	20 20 20	80 160 240
	3 3 3	1 2 3
	Each one gets 143	$240 + 240 = 480$

Figure 7: Use of significant mathematical ideas

Place value has traditionally been viewed as a prerequisite for learning computation. Under an invented strategies approach, however, students learn place value ideas as they make sense of computational examples. Additionally, they appear to develop a stronger, more connected understanding of place value in the course of pursuing the purposeful goal of solving a problem. This point is made by Carpenter, Fennema, Franke and Empson (1999): "Problems with two- and three-digit numbers actually provide a context for children to develop an understanding of base-ten numbers" (p. 64). This finding offers a new approach to teaching place value.

Once again, the central and active role of the teacher in facilitating learning is highlighted. Teachers need to recognise the mathematics, highlight it with students and engage them in thoughtful discussions about it.

6. Mathematical tools support students' learning

The notion of mathematical tools is far broader than just physical materials and includes the use of language and written symbols (Hiebert et al., 1997). In addition to language and symbols, hundreds charts and base-ten blocks are particularly valuable tools for building understanding of addition and subtraction strategies. The hundreds chart can help children compute by starting with a number and counting on or back by tens and ones (Figure 8a). Modelling a problem with base ten blocks highlights partitioning of numbers and grouping by tens (Figure 8b).

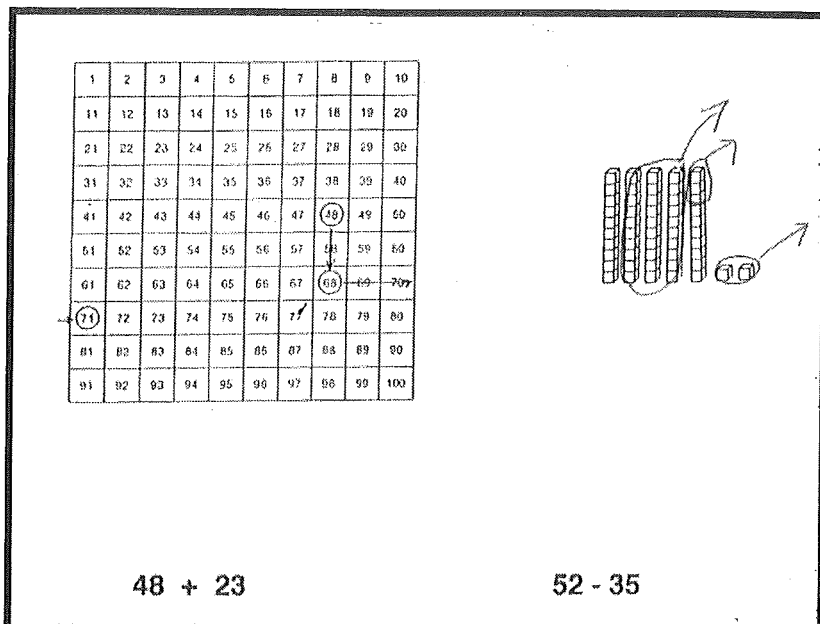


Figure 8: Use of hundreds charts and base-ten blocks

Teachers initially provided opportunity for students to explore and construct their own meanings for both tools. Students then used them in ways that made sense to them. This is quite different from requiring them to follow a prescribed series of steps in the use of the tools. A variety of methods evolved and the ensuing discussions enriched all students.

These tools were helpful to all students in some way—in helping them construct strategies, in promoting understanding of other students' strategies, in allowing them to extend their repertoire, in developing place value, and in providing a primary means of computing initially.

Developing Written Algorithms

The issue of traditional written algorithms is a contentious one. Some argue that they are inherently harmful to student understanding and should be excluded from the curriculum (Kamii & Lewis, 1993), while others suggest that they have an important role in students' mathematical development and that some level of work should be included (Addington, 1996). The antipathy of some reformers is counterbalanced by the belief of the general public that efficient computation (the algorithms they learned) is highly important. Gravemeijer et al. (*this volume*) take the position that specific algorithms need to be directly taught, but ones that are different from the standard ones. In general, teachers are uncertain regarding what they should do about written computation and how they should do it. They seek pragmatic guidance.

We concur with Hiebert et al. (1997, p. 6) that "the primary goal of mathematics instruction is conceptual understanding" and also agree that "... setting conceptual understanding as the primary goal does not mean ignoring computation skills". But discussions about skills must address the issues of efficient written computation and the

complexity of the work. Unfortunately, while we have very clear (and negative) images of traditional teaching of computation, we have little evidence of what *can* occur in sense-making classrooms.

In this section we address the experiences of teachers across several grade levels with standard computation when the instructional emphasis is on understanding mathematics and extended work with invented algorithms. The discussion provides another look at how traditional computation *can* be developed. It, however, is *not* an argument that traditional algorithms *must* be part of mathematics programs.

We will now share two sets of observations about the learning of traditional computation. First, we address what students bring to the work. We note that students:

- believe all mathematics will make sense and see computation as another context in which to investigate mathematics;
- are aware of and curious about traditional algorithms;
- are somewhat familiar with traditional algorithms from previous occasions when they have been discussed;
- are interested in exploring new strategies because of their confidence with ones they have learned; and
- have developed a deep understanding of place value from work with invented algorithms and are able to connect this knowledge to traditional algorithms (see Figure 9).

40 + 20 = 60	48	
8 + 7 = 15	<u>+ 27</u>	¹ 48
60 + 10 = 70	60	<u>+27</u>
70 + 5 = 75	<u>15</u>	<u>75</u>
	<u>75</u>	

Figure 9: Three written records for 48 + 27

Second, there are several key points that characterise the work on traditional algorithms in the project classrooms.

1. Systematic attention does not occur until students have had extensive experience with invented algorithms. The timing is determined by teachers based on their assessment that students are ready for the work.
2. The work may begin with the teacher conducting a more in-depth discussion of the traditional algorithm after it has been suggested by a student; she may do this by asking questions that have students compare it to one or more algorithms that already have been presented. Earlier discussions have focused on efficient versus less efficient algorithms and discussions on the traditional algorithm can be related to the broader discussion of efficient procedures. Students find the ideas in traditional algorithms are not new, even though the way the work is recorded is different.

3. Instruction occurs the same way that all work with invented algorithms occurs: students work together; various materials are available; and there is much whole group discussion.
4. Following the class discussion, students try the algorithm with four or five examples. This may be done for a few days. During this time errors or misconceptions are discussed. Students are also encouraged to solve examples using at least one additional strategy. They also may discuss when one approach is preferable to the other. Effort is made to ensure that students view it as *one* approach, not the *only* approach.
5. Just as students developed proficiency with other strategies over time, proficiency with the new strategy proceeds in the same manner.

Overall, the work with traditional algorithms proceeds smoothly. It does not consume a great deal of class time, and it does not involve large amounts of drill. There are few of the usual errors and misconceptions; and this is due to the students' understanding of the underlying ideas and their expectation that answers must make sense.

A Classroom Vignette

The following vignette illustrates one way that traditional algorithms develop. It describes the way multiplication developed in a fourth grade class (Pottebaum, 1999) of 32 students who represented various ethnic backgrounds and a mix of both special needs and high achieving students. The teacher's primary goal was to help her students develop a strong conceptual understanding and insight into computational strategies. Her district also expected her to teach traditional algorithms. Her concern was to do both in ways that would make sense to her students.

Early in the year, six weeks was spent developing multiplication and division concepts, with a strong emphasis on multiples, arrays, strategies and solving various problem types. Students also found products for clusters of related multiplication sentences (see Figure 10).

$$2 \times 6 = 12$$

$$4 \times 6 = 24$$

$$10 \times 6 = \underline{\quad}$$

$$40 \times 6 = \underline{\quad}$$

$$42 \times 6 = \underline{\quad}$$

Figure 10: Related multiplication sentences

Following the unit, the teacher included one or two multiplication problems on her daily review for which they were encouraged to use their own approaches. For a problem involving 4×32 several strategies emerged. One student used the following number sentences: $2 \times 4 = 8$, $30 \times 4 = 120$, $120 + 8 = 128$; while another student

demonstrated the traditional algorithm, which led to a discussion of the relationship between the approaches. After a few weeks, the students used both the invented and traditional algorithm with understanding.

Later this work was extended when the teacher created a problem about 32 students doing 25 sit-ups. Among the student strategies presented were firstly, the use of four partial products (20×30 , 20×2 , 5×30 , and 5×2); secondly the use of two partial products (30×25 and 2×25); and thirdly the traditional algorithm. Some students initially struggled with how to find the product. Over several weeks, the students solved problems of the same type. Eventually, most students used both the traditional algorithm and at least one other strategy.

The teacher was delighted, as her belief was reinforced that instruction can lead to *both* understanding and skill. She also noted that the emphasis on computation remained within her overall emphasis on understanding mathematics and solving problems.

Summary

This chapter has examined the learning of computation in classrooms that stress mathematical understanding, under instruction that takes students' thinking seriously and builds on it, and with a curriculum in which students have ongoing experiences with major ideas and skills. A number-sense approach to computation has been presented in which computation is an important site for expanding students' number sense and developing computational strategies. We have shown that an ongoing approach to computational strategies results in efficient and useful mental and written algorithms, and also leads naturally to traditional written algorithms.

References

- Addington, S (1996). *Our methodology is obsolete: A response to 'Is long division obsolete?'* *Mathematics education dialogues*. 16. Reston, VA: NCTM.
- Carpenter, T.P., Fennema, E., Franke, M.L., Levi, L. & Empson, S.B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.
- Hiebert, J., Carpenter, T.P., Fennema, E., Fuson, K.C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- Kamii, C. & Lewis, B.A. (1993). The harmful effects of algorithms in primary arithmetic. *Teaching K-8*, 23, 36-38.
- Pottebaum, P. (1999). Concepts, skills, and so much more. *Iowa Council of Teachers of Mathematics Bulletin*. Winter, 32-39.
- Resnick, L.B. (1995). Inventing Arithmetic: Making children's intuition work in school, In C.A. Nelson (Ed.). *Basic and applied perspectives on learning, cognition, and development* (pp. 75-101). Hillsdale, NJ: Erlbaum.
- Resnick, L.B., Bill, V.L., Lesgold, S.B., & Leer, M.N. (1991). Thinking in arithmetic class, In B. Means, C. Chelemer, & M.S. Knapp (Eds.). *Teaching advanced skills to at-risk students: Views from research and practice* (pp. 27-53). San Francisco: Jossey-Bass.

- Reys, B.J. & Reys, R.E. (1998). Computation in the elementary curriculum: Shifting the emphasis. *Teaching children mathematics*, 5(4), 236-241.
- Trafton, P.R. & Thiessen, D. (1999). *Children learning through problems: Number sense and computational strategies*. Portsmouth, NH: Heinemann.
- Szieska, K. (1995). *Math curse*. Illustrated by L. Smith. New York: Viking.

Semi-informal Routines as Alternatives for Standard Algorithms in the Primary School

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Abstract

There is a growing opposition to taking the mastery of the conventional written algorithms for the four basic operations as a central goal in mathematics education in primary school. The opposition is based on the diminishing practical importance of those written algorithms, and on the growing awareness of the limitations of the common way algorithms are taught. At the same time, goals like mathematical reasoning, communicating and problem solving get more emphasis. One may wonder, however, if it is wise to do away with the conventional algorithms, without replacing them with something else. In this chapter, helping the students develop semi-informal routines is proposed as an alternative for teaching conventional algorithms in a ready-made form. It is argued that these routines should be grounded in well-developed number sense. A critical factor in the approach is that applied problem situations are modelled in such a manner that these models can be used to support the students' informal situation-specific solution strategies. These models then help the students to structure their way of working, and this lays the basis for flexible routines that can serve as alternatives for conventional algorithms. At the same time, however, these routines can be developed into conventional algorithms later, if one wishes to do so.

Introduction

For a long time, algorithms, especially the written algorithms for the four basic operations, have formed the core of the arithmetic curriculum for primary school. Nowadays, however, the self-evidence of teaching those algorithms is under discussion. An important factor is the wide availability of pocket calculators, but there is also a shift in thinking about the goals of mathematics education. The latter shift encompasses a

reduced emphasis on traditional ‘basic skills’ and an increased emphasis on mathematical reasoning, communicating, and problem solving. We may refer to the American ‘NCTM Standards’, for instance, as a clear exponent of this shift.

In the following we will elaborate this change in perspective, and we will argue for having strategies, routines and algorithms emerge from a sound foundation of number sense. We will describe some informal routines that can be developed as alternatives for the conventional written algorithms. We will use the term ‘informal’ rather loosely to indicate methods that are at least in part invented by the students, and often reflect characteristics of the situation. In contrast, formal methods would be methods that have a formal character in the sense that they are tied to conventions, and basically cannot be adjusted to the characteristics of the situation. The informal routines described here include the use of the empty number line for two-digit addition and subtraction; the Candy-Factory scenario for multi-digit addition and subtraction; and the ratio table for solving multiplication and division problems.

Goals for Mathematics Education

Until not very long ago, a flawless mastery of written algorithms was a skill that would clearly enhance one’s employability. With the availability of calculators and computers the importance of these skills for everyday-life situations is being questioned. In fact, we have to acknowledge that most of us do not use the algorithms learned at school when we have to solve everyday-life practical problems. Take, for instance the following question:

How many chocolate bars costing \$0.75 each could one buy for \$ 30.00?

Some may reason: twice 0.75 would be 1.50, four would be 3.00, thus forty bars would cost \$30. Others might realise that $0.75 = \frac{3}{4}$, and use this to support the above strategy. Taking $\frac{3}{4}$ as a starting point could also support the idea of thinking of a ratio of three to four; the numbers of dollars is to the number of chocolate bars as 3 : 4. The chance that anyone would solve this problem by going through the algorithm for long division would be pretty slim—outside school.

When looking at the informal solution methods, we may wonder if one could speak of a conscious application of strategies. It does not seem plausible. Instead, we may infer that people are guided by knowledge of number relations like: $0.75 = \frac{3}{4}$, $2 \times 0.75 = 1.50$, $2 \times \frac{3}{4} = 1\frac{1}{2}$, or $10 \times 3 = 30$. This may include knowing that a number ending in .5 or .75 is easy to double.

Thus, if our goal in mathematics education in primary school would be to help students develop the kind of solution methods they will actually use in everyday-life, an emphasis would have to be put on developing number relations, or more general: on developing number sense. What this means for instruction may be elucidated by taking the so-called ‘basic facts’ for addition and subtraction up to twenty as an example.

Of course, students do not have to know all those basic facts by heart. Many unknown number facts can easily be derived from known number facts. For example, the number fact ‘ $6 + 7 = 13$ ’ can be derived in the following ways:

$$\begin{array}{ll}
 6 + 7: & 6 + 4 = 10, 10 + 3 = 13; \\
 6 + 7: & 6 + 6 = 12, 12 + 1 = 13; \\
 6 + 7: & 7 + 7 = 14, 14 - 1 = 13; \\
 6 + 7: & 7 + 3 = 10, 10 + 3 = 13; \\
 6 + 7: & 6 + 7 = 5 + 1 + 5 + 2 = 10 + 3 = 13.
 \end{array}$$

These solution methods are often described as the application of general strategies, like filling up the ten; looking for the nearest double; using the commutative property; and splitting off fives. However, what we see as the application of strategies does not have to be experienced as applying strategies by the students. Instead of using strategies, the students may be using their knowledge of number facts. They might, for instance, be so familiar with relations like $3 + 3 = 6$, and $7 + 3 = 10$, that $7 + 3 = 10$, $10 + 3 = 13$ presents itself as an obvious solution.

We want to stress this point since it exemplifies our view on mathematics education. We want to base mathematics education in the students' thinking. In our view, students should build up on what they know. That does not mean that strategies and algorithms are odious, but it means that strategies and algorithms should emerge from the students' own activity. This can be contrasted with presenting efficient procedures to the students in a ready-made form. The problem with ready-made procedures is that the students can easily apply them without understanding, and without showing their lack of understanding. Students, then, may experience mathematics as a topic that is not built on understanding, at least not for them. Moreover, if we do not explicitly try to keep the students' perspective up front, we may unknowingly take things for granted that are crystal clear for us but not for the students. The standard procedure of filling up the ten may be an example. Kraemer (personal communication, August, 1998) noted that many weaker students do not experience 'ten' as being any different than 'nine' or 'eleven', at the age that the filling-up ten procedure is presented to them. For these students, this procedure will not, and actually cannot, make much sense.

The alternative is, in our view, to foster a process by which such procedures emerge from the students' own activities. In this line of thinking, strategies can be thought of as generalisations over various individual cases. Having solved numerous tasks like ' $6 + 7 = ?$ ', the students might start to see that solution methods can be grouped by characteristics like 'using a nearby double', or 'filling up the ten'. After having made these different types of solution methods explicit, the students may start thinking about them as strategies to be applied. One of them might even be developed into a standard procedure that functions as an algorithm that can be applied with insight. This should not be our main objective, however.

What we want to stress here is the value and usefulness of a rich basis in terms of a framework of number relations that can be used flexibly. In relation to this, we may refer to Greeno's (1991) use of the metaphor of an environment to describe number sense; an environment, where one knows one's way, including short-cuts, landmarks, and so forth. We may note, that these 'environments' will differ from student to student. Each student will have his or her own set of more and less familiar number relations. This could be seen as problematic if the objective was to ensure that all students would use 'optimal' solution methods for various types of tasks. In contrast, our objective is that the students use what they know in a flexible manner. In other words, we want the students to use the elements of their idiosyncratic set of number relations as building

blocks for solution methods that make sense to them. In spite of this idiosyncrasy, as educators, we will favour the development of certain number relations over others. We will try to foster those number relations, which we know provide footholds for flexible arithmetic and estimation. We will try to do this, of course, in an indirect manner; by carefully choosing the numbers in the tasks that are posed, and by including in these tasks, connections with numbers that play key roles in everyday-life situations.

Alternatives for Two-digit Addition and Subtraction

The alternative approach to algorithms we are proposing is to start with the informal solution methods of the students and help them to develop those into flexible routines. As we will show in the following, this comprises the use of various models that support semi-informal routines, which, in principle, can be developed into the standard conventional algorithms, if one would want to do so.

As an introduction, think for a moment about what variety of solution methods you would expect grade-three students to come up with, when the following task is posed to them.

*A book has 64 pages. You have already read 37 pages.
How many pages still have to be read?*

An investigation of 43 Dutch students at the end of grade 3 revealed the solution procedures shown in Figure 1 (Ontwikkelgroep Speerpunt Rekenen, 1991). The results show that there is a wide variety of solution procedures. We might categorise these solution procedures as:

- string method; adding on in steps, or taking away in steps,
- splitting; splitting ten and ones separately and combine the results afterwards,
- skillful reckoning; using the characteristics of the numbers involved, and
- column algorithm; mental execution of the column algorithm.

The results show that the string method is both popular and effective. At the time of the investigation, however, this method was not taught in schools—it was a strategy that the students had invented by themselves. What was taught in schools—often with MAB, or Dienes blocks—was a splitting approach. The rule-governed approach of MAB blocks does not leave much room for informal strategies, but children were using these nevertheless. One of the factors must have been that the book problem is a different type of task. We can make a distinction between set-type situations and linear-type, or counting-type situations. The book problem can be classed as counting-type or linear, since the page numbering refers to the number row and connects with counting on and counting back. In block tasks, on the contrary, the emphasis is on structuring quantities in groups. For example, 85 is structured in 8 groups of ten and 5 ones. Given the spontaneous use of string methods, one might want to look for ways of supporting and elaborating this approach. This can be done by employing the empty number line as a model (Whitney, 1988; Treffers & de Moor, 1990).

A book has 64 pages. You have already read 37 pages.
How many pages still have to be read? [N = 43]

Correct [N=34]

37+20,	57+7=64,		27 added	[8]
37+10=47,	47+10=57,	57+7=64,	27 added	[4]
37+3=40,	40+20=60,	60+4=64,	27 added	[4]
37+3=40,	40+4=44,	44+20=64,	27 added	[1]
37+3=40,	40+24=64,		27 added	[1]
7+7=14,	14+50=64,	so 37+7+20=64,	27 added	[1]
64-4=60,	60-20=40,	40-3=37	27 taken away,	[2]
64-10=54,	54-10=44,	44-7=37,	27 taken away	[1]
60-30=30,	7-4=3,	30-3=27		[1]
37-30=67,	67-3=64,		27 in all	[2]
64-30=34,	34-10=24,	24+3=27		[1]
64-10=54,	54+3=57,	57-20=37,	27 taken away	[1]
60-30=30,	34-7=27,			[4]
7+7=14,	10+30=40,	40+20=60,	60+4=64	[1]
30+30=60,	7+7=14,	14 > 10, so change one 30 into 20 -> 37+27=64		[1]
First complement to 14, then complement to 6(0); this gives 27				[1]

Incorrect [N=9]

60-30=30,	4-7=0	answer: 33	[1]
60-30=30,	4-7=3	answer: 33	[1]
Half of 60 is 30 and 4+3=7		answer: 33	[1]
Half of 60 is 30, 7-4=3		answer: 33	[1]
Written algorithm		answer: 22	[1]
30+20=50, add 7, complement to get 64		answer: 25	[1]
37+3=40, complete the ten, that takes 3,		answer: 23	[1]
4,5,6 to get 60, next the 3 and the 1, because of 64 and 37,			
that is simply 3+1=4,		answer: 34	[1]
Tally marks		answer: 25	[1]

Figure 1: Grade 3 solution procedures (from Ontwikkelgroep Speerpunt Rekenen, 1991)

String methods can be symbolised on an empty number line by marking the numbers involved, and drawing the 'jumps' that correspond with the partial calculations. For example, solving $64 - 29$ by subtracting 4, 10, 10, and 5 respectively would result in an inscription as shown in Figure 2.

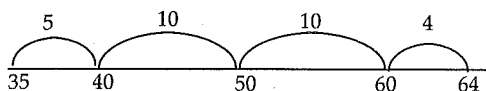


Figure 2: $64 - 29$ on the empty number line

Apart from functioning as a means of describing solution methods, the number line also supports the execution of counting-type methods, by offering a way of scaffolding of both partial calculations and partial results. Furthermore, the empty number line can be used to depict more sophisticated strategies, like compensating. For example, $95 - 19$ can be solved by first subtracting 20 from 95 and then adding one. The empty number line can be used to explain and justify this strategy (see Figure 3).

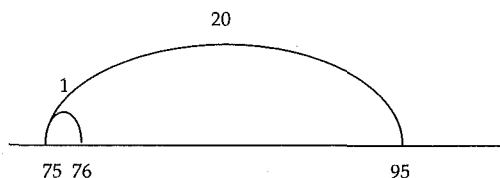


Figure 3: Compensating to solve $95 - 19$

The number line, in short, can play a role in supporting the elaboration of the students' informal strategies, and the development of more sophisticated ones. If we compare using the number line in this manner, with working with the blocks in a rule-governed manner, we see a clear advantage for the number line: The students can adapt the model to their thinking. In the case of the empty number line, the model is not employed to steer the student's thinking, instead, the model is meant to be adapted by the students to fit *their* thinking. The student decides what steps to make, and marks the number line accordingly.

We may note that most of the number relations we have discussed up to now, revolved around our decimal numeration system. However, we want to promote a much richer framework of number relations. Think, for instance, of a task like ' $72 - 38 = ?$ '. One could, of course, solve this task with a string method, but there are other options. One might, for instance, think of ' $72 = 2 \times 36$ ', and conclude that ' $72 - 38 = 34$ '; '2 less than 36'. Next to this type of number relation that encompasses multiplication facts, or doubling and halving and so forth, we would also want to make connections to the role numbers play in everyday life.

This implies looking at numbers in a different way. If we take a number like 52, for instance, it would not be sufficient to know the quantitative meaning of 52, and to know how 52 is constituted of tens and ones. The students should also have a sense of what 52 metres is, or 52 years. They should be familiar with 52 as $2 \times 25 + 2$; two quarters and two pennies, or with 52 as 4×13 (weeks in a year); or with 52% as a little bit more than one half. We want to stress that the connection with real-life, especially with measurement and money, can also support the development of a framework of useful number relations—in particular for number relations that can be used in estimations.

Alternatives for Column Algorithms in Addition and Subtraction

For multi-digit addition and subtraction we want to discuss the so-called Candy Factory scenario (McClain, Cobb, & Bowers, 1998). Superficially the materials used in the Candy Factory scenario seem to have much in common with Dienes blocks, but in contrast with the top-down approach of Dienes blocks, a bottom up approach is

followed, in which students themselves create a model for the operations. A similar approach was suggested by Dekker, ter Heege and Treffers (no year) with a scenario involving packing the golden coins of a sultan.

In the Candy Factory scenario, the students are told about a factory that produces candies. One of the first problems the students are confronted with, concerns the manner of packing the candies. They are told that the candies are to be packed in rolls, and the question is what size should those rolls be. After some discussion, in which ‘packing in rolls of ten’ is usually proposed by a number of students, the student are told that the management of the factory decides to pack the candies in rolls of ten, ten rolls in a box, and ten boxes in a crate. In this way, ‘base ten’ is introduced as a conscious choice, which can be argued about.

In the next phase of the story, the students act as workers in the storeroom, who have to pack fresh produced candies, and if needed unpack candies to match a given order to be sent out. In the process, the students develop drawings to keep track of these transformations. On the basis of these drawings, inscriptions using dots for candies, lines for bars, and squares for boxes are developed as a convention. While the story evolves, the students get the task of keeping track of the number of candies that vary under the influence of production and sale of candies. In relation to this an inventory form is introduced on which one can keep track of the number of candies in the storeroom, but that at the same time it can be used as a means of scaffolding the transformations involved (see Figure 4).

Crates	Boxes	Rolls	Candies
	5	7	3
	+1	+5	+8
	6	12	11
	6	13	1
	7	3	1

Figure 4: The Candy Factory inventory form

In this manner, a setting very similar to working with Dienes blocks is created. However, there is a significant difference with a top-down approach, for the notations in the Candy Factory scenario are meaningful to the students because of the history of these notations.

Another difference is that in such an alternative set up, the students are free to develop their own solution methods. We may illustrate this with some examples from a third-grade teaching experiment in the Netherlands (Boswinkel, 1995). The students were given the bare example ‘399 – 174’.

Anne splits the numbers in hundreds, tens, and ones. She writes:

399
174 –
300 – 100 = 200
90 – 70 = 20
9 – 4 = 5
together: 225

Another student, Nazia, first simplifies the task. She argues: “As a present, I add 1 to 399 for a while”. Then she subtracts: $400 - 100 = 300$; $300 - 70 = 230$; and $230 - 4 = 226$. She concludes with: “Then, since I have added that 1, I have to subtract it: $226 - 1 = 225$ ”.

Danny solves the problem by supplementing: “First, 5 added to 74, makes 79, then adding another 20, which is 25 together, then from 100 to 300, which is 225 altogether”.

Virgil breaks the task down into a series of subtractions:

$$\begin{aligned} 399 - 100 &= 299 \\ 299 - 70 &= 229 \\ 229 - 4 &= 225. \end{aligned}$$

Anne's method	Nazia's method	Virgil's method
399	399 (+1)	399
<u>174</u> -	<u>174</u> -	<u>174</u> -
200	300	299
20	230	229
<u>5</u>	226 (-1)	225
225	225	

Figure 5: Informal procedures in column format

Most of these methods can be written in some sort of column-type format as shown in Figure. 5, and similar methods can be developed for more complex tasks (see Figure 6). As such, these informal methods can become the basis for semi-informal algorithms that may bear idiosyncratic characteristics reflecting their history, or that may be varied depending on the numbers involved. If deemed necessary, these semi-informal algorithms can easily be transformed into the conventional algorithms. We think it would be regrettable, however, if algorithms were to completely drive out the flexible use of number sense, which is demonstrated in the above examples.

745	745
<u>462</u> -	<u>462</u> -
300	300
-20	280
<u>3</u>	283
283	

Figure 6: Informal procedures for 745 – 462

Multiplication and Division

Most children do not simply memorise the multiplication tables in order to know them by heart—rather, they use the multiplication facts they know to derive more

difficult ones. According to Ter Heege (1985) the solution methods can be categorised in terms of the following strategies:

- changing order [$3 \times 6 = 18$, because $6 \times 3 = 18$]
- doubling [$2 \times 6 = 12$, therefore $4 \times 6 = 24$]
- halving [$10 \times 6 = 60$, therefore $5 \times 6 = 30$]
- adding [$2 \times 6 = 12$, therefore $3 \times 6 = 18$]
- subtracting [$10 \times 6 = 60$, therefore $9 \times 6 = 54$]

As before, however, we would like to stress that children do not have to experience their solution as a conscious application of a strategy. In their view, they may just use what they know about certain numbers.

Solution methods similar to these may be observed when students are solving problems in the context of recipes (cf. van Galen & Wijers, 1997). In the task shown in figure 7, the students are asked to find out what ingredients are needed for 2, 8, 6, 10, or 16 servings when a recipe for 4 servings is given. This situation seems to lend itself to combining data of one or more columns to calculate the next. For example, you know that four servings take $\frac{1}{2}$ cups of flour, so two servings take half of $\frac{3}{4}$, which equals $\frac{3}{8}$, and six servings take the same amount as $4 + 2$ servings, which is $\frac{3}{4} + \frac{3}{8} = \frac{9}{8}$, or $1\frac{1}{8}$.

Servings	4	2	8	6	10	16
Flour (cups)	$\frac{3}{4}$					
Margarine (cups)	$\frac{1}{4}$					
Powdered Sugar (tablespoons)	3					
Water (teaspoons)	$2\frac{1}{2}$					
Yogurt (cups)	$1\frac{1}{3}$					

Figure 7: Recipes task

The use of a ratio table can easily be expanded to include the multiplication of integers (cf. van Galen & van den Heuvel-Panhuizen, 1997). Using a ratio table, the multiplication 18×23 for instance, can be solved in various ways (see Figure 8).

1	2	4	8	16	18
23	46	92	184	368	414

1	2	20	18
23	46	460	414

1	10	8	18
23	230	184	414

Figure 8: Various ways of calculating 18×23 with a ratio table

A neat feature of the ratio table is its flexibility, which leaves room for a variety of solution methods. However, if one wishes to, a standard procedure can be developed that is based on the characteristics of our decimal system, and resembles the conventional algorithm (see Figure 9).

152×242

1	2	50	100	152
242	484	12100	24200	36784

242
<u>52</u> ×
484
12100
<u>24200</u>
36784

Figure 9: Ratio table and standard algorithm

Discussion

Algorithms are losing their central position in mathematics education in primary school. Some would like to abandon them completely, because of their association with drill and practice. One cannot, however, do away with the standard algorithms for the four basic operations without replacing them with some alternative. In this chapter, we showed how students could develop semi-informal routes that can function as flexible alternatives for the conventional algorithms. We want to stress, however, that in our view, there is nothing wrong with algorithms as such. In a wider sense, without limiting ourselves to arithmetic, algorithms are an essential part of mathematics. On the one hand, they can be seen as an end product of a process of mathematising; on the other hand, they can function as the tools to attack more complex problems. The issue here, however, is that the conventional written algorithms for addition, subtraction,

multiplication and division can easily be replaced by the technology of calculators, and computers. We therefore question the need for mastering these conventional algorithms, although we do see value in experiencing the reinvention of those algorithms.

Although we think technology has diminished the need for algorithms, we still do see a significant role for flexible semi-informal arithmetical routines. Our age is often referred to as the “information age”, but a large part of this information is numerical. To be able to come to grips with this kind of information, students will have to be able to interpret and work with numbers in a flexible and insightful manner. We therefore plead for the development of number sense. In this context, semi-informal routines can be seen as a natural extension of a framework of number relations that is linked with the role number plays in the reality outside school. We argue for an instructional approach in which the students build up on what they know, instead of trying to come to grips with strategies and procedures that are presented to them in a ready-made form. We do think that this is the best way to avoid inappropriate solution methods that students may not fully understand. We also consider it important that students experience the process of learning mathematics as constructing mathematics. Finally, we think that such an approach is the best guarantee for the construction of a wide variety of routines that can be flexibly tailored to the characteristics of the task at hand.

References

- Boswinkel, N. (1995). Interactie, een uitdaging, *Tijdschrift voor nascholing en onderzoek van het rekenwiskundeonderwijs*, Panama Post, 14(1), 4-13.
- Dekker, A., ter Heege, H., & Treffers, A. (no year). *Cijferend vermenigvuldigen en delen volgens wiskobas*. Utrecht: OW & OC.
- Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, 22, 170-218.
- McClain, K., Cobb, P., & Bowers, J. (1998). A contextual investigation of three-digit addition and subtraction. In L.J. Morrow & M.J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Ter Heege, H. (1985). The acquisition of basic multiplication skills. *Educational Studies in Mathematics*, 16, 375-388.
- Treffers, A. & de Moor, E. (1990). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs, deel 2*. Tilburg: Zwijsen.
- Ontwikkelgroep Speerpunt Rekenen, Vuurmans, A.C. (Ed.), (1991). *Rekenen tot honderd*. KPC, Hertogenbosch.
- van Galen, F. & van den Heuvel-Panhuizen, M. (1997). Number tools. In National Center for Research in Mathematical Sciences Education & Freudenthal Institute (Eds), *Mathematics in context, A connected curriculum for grades 5-8*. Chicago: Encyclopedia Britannica Educational Corporation.
- van Galen, F. & Wijers, M. (1997). Some of the parts. In National Center for Research in Mathematical Sciences Education & Freudenthal Institute (Eds.), *Mathematics in context, A connected curriculum for grades 5-8*. Chicago: Encyclopedia Britannica Educational Corporation.
- Whitney, H. (1988). *Mathematical reasoning: Early grades*. Princeton (unpublished paper).

Rethinking Percent Instruction in the Middle Grades

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Percent is an integral element of Western society. In the real world, percent usage abounds as discounts, profits, losses, savings, increases, and statistics; and features in advertisements, shops, newspapers, magazines, and various other media. Because of its real-world application, percent is an important topic within the school mathematics curriculum, featuring prominently in the middle years curriculum (upper primary through to secondary school). But percent is a notoriously difficult topic both to teach and to learn (Cole & Weissenfluh, 1974; Smart, 1980), and extensive research into students' performance on percent tasks indicates a long history of student failure and frustration (Parker & Leinhardt, 1995). In this chapter, percent instruction is considered in light of the many approaches that instruction in this topic can take. The focus of this chapter is to present a case for a streamlined approach to percent instruction that promotes students' understanding of the proportional nature of percent situations. In the first part of this chapter, students' performance on percent tasks and various approaches to percent instruction are described, and the fundamental meaning of percent as a proportion is outlined. In the second part of this chapter, an efficient and effective four-step method for percent problem solving that enables students to represent and solve percent problems proportionally is presented. An argument is made for the selective algorithmic component of the method, which is seen to facilitate successful operation in the domain of percent, enabling immersion in and exploration of the whole conceptual field of percent usage. The method is discussed in terms of its potential to take percent instruction beyond practise of mindless written computational procedures, to building students' knowledge of percent as a proportion, promoting number sense and estimation skills, and fostering rich percent schema development.

Students' Percent Performance

Measures of students' performance on percent tasks paint a grim picture of poor performance. In their summary of research spanning six decades, Parker and Leinhardt (1995) concluded that "percent is a topic in which students have displayed inadequate performance, and in some cases, utter confusion, for over 60 years" (p. 422). More specifically, results of the fourth National Assessment of Educational Performance (NAEP) in mathematics (Kouba, Brown, Carpenter, Lindquist, Silver, & Swafford,

1988) provided evidence that students at the seventh-grade level in this assessment had difficulty with percent calculations, and also appeared to lack understanding of percent concepts underlying the calculations. Results also indicated that absence of conceptual understanding and inability to apply percent knowledge in problem situations was a trend that continued through to students in the eleventh grade. For example, only 32% of seventh-graders and 62% of eleventh-graders could calculate 4% of 75. Further, only 9% of seventh-grade and 37% of eleventh-grade students could solve a two-step word problem involving simple interest calculations.

In a more recent study with eighth-grade students (Dole, 1999a), similar results were found. Of the 117 students presented with similar items, 57% could calculate 4% of 75, but only 5% could solve a similar two-step word problem. On a more positive note, other research studies have indicated that students in the middle school (fifth graders through to eighth graders) have developed a basic conceptual understanding of percent as a base of 100, and that they can competently apply percent benchmarks of 50% and 25% (Dole, 1999a; Dole, Cooper, Baturo, & Conoplia, 1997; Gay & Aichele, 1997; Lembke & Reys, 1994). Research has also indicated that, before formal percent instruction, students use a variety of intuitive strategies to solve (simple) percent problems (Lembke & Reys, 1994), and students who can successfully operate in the domain of percent draw upon well-developed problem solving and estimation skills to check the reasonableness of calculations; that is, they draw upon number sense (Dole et al., 1997). However, as reported by Dole et al. (1997), such number sense applied to percent was evident in only very few students at the eighth, ninth and tenth grades tested in this study. In contrast, low ability students in this study were seen to rely on the use of key words or formulae to solve percent problems; they did not naturally check reasonableness of solutions, and they could not proceed if they could not access a formula. A significant finding of this study was that students who operated successfully in the domain of percent did so, not because of their percent knowledge, but because of their high level of number sense. That is, through trial and error, they accessed procedures that led to correct solutions rather than drawing on percent schema knowledge.

Research into students' percent performance not only highlights students' difficulty in developing meaning in percent situations, but also that as a consequence of school instruction in percent, students' intuitive percent understandings and flexibility in thinking gives way to mindless application of rules and procedures (Gay & Aichele, 1997; Lembke & Reys, 1994; Parker & Leinhardt, 1995). As stated by Parker and Leinhardt (1995):

...percent has become entangled in the mesh of conversion rules for changing decimals to fractions, fractions to decimals, improper fractions to mixed numbers, and mixed numbers to improper fractions. The emphasis in percent is not what it is but how to compute it quickly. (p. 434)

For all students, then, there is a critical need to focus percent instruction to develop students' knowledge of percent as a rich domain in its own right, as well as to build on number sense to promote percent sense to ensure competent and meaningful percent performance.

Percent Instruction

The literature is replete with many and varied suggestions for teaching percent, and these typically link percent to other rational number topics, including fractions, decimals, ratio, and proportion. Loosely, such approaches can be classified as either methods designed to (i) develop conceptual understanding, or (ii) assist in solving percent applications, either conceptually or procedurally. For developing the part/whole concept of percent, the representation of percentages on 10 x 10 grids is suggested (e.g., Bennett & Nelson, 1994; Cooper & Irons, 1987; Reys, Suydam, & Lindquist, 1992; Van de Walle, 1998), to explicitly link the meaning of percent as parts per hundred and also to show fraction-percent equivalence, particularly for fractions that can assist mental computation of percentages of quantities (e.g., 50% as $\frac{1}{2}$, 25% as $\frac{1}{4}$, 10% as $\frac{1}{10}$). Other suggestions for developing conceptual understanding include focusing on the language of percent situations through investigating the special language of percent used in society (e.g., the use of such terms as 100% attendance; 110% effort) (Glatzer, 1984); interpreting percent situations as ratios (Brown & Kinney, 1973); studying percent expressions as statements of proportion (Schmalz, 1977); as well as building estimation skills through explorations of patterns of simple ratios (Cooper & Irons, 1987; Glatzer, 1984).

As students move to secondary school, percent instruction (particularly in the eighth grade) tends to focus on procedures for calculating the three types of percent problems found in school mathematics textbooks, commonly referred to as the 'three percent problems' by junior high school teachers (Van de Walle, 1998) as follows:

- (i) finding a part or percent of a number (e.g., what is 28% of 153?);
- (ii) finding a part or percent one number is of another (e.g., express 56 out of 74 as a percent);
- (iii) finding a total amount when a certain part or percent of that amount is known (27 is 78% of what number?).

In the literature, a diverse range of methods and strategies for assisting computation in percent is suggested, and these can be found to rely on a range of skills and procedures, including whole number, decimal, and fraction multiplication and division; fraction, decimal and percent conversions; equivalent fractions; and proportion equations. Suggestions include the use of concrete materials and diagrams, such as fraction/percent overlays and elastic strips (e.g., Weibe, 1986); 10 x 10 grids (Bennett & Nelson, 1994), comparison scales (Dewar, 1984; Haubner, 1992), as well as strategies for identifying key words and mnemonic devices (Boling, 1985; McGivney & Nitschke, 1988). Combined with the diversity of methods for percent calculation, analysis of middle school mathematics textbooks reveals a further range of methods. Parker and Leinhardt (1995) described five such approaches typically found in schools that can be labelled as traditional/case, formula, equation, proportion, and unitary methods. According to Smart (1980), the selection of a particular computational procedure studied by students is dependent upon the teacher's personal bias towards a particular method.

Beyond procedures for the three percent problems is the need to provide students with skills to analyse the accuracy of percent usage in the real world. Misapplications of percent reflect a lack of conceptual understanding of appropriate percent usages in common situations. The following four examples of percent usage in the real world, compiled by Tout and Johnston (1995) show misuse of percent, which at first glance appear feasible:

(1) A quote in a newspaper

Ten years on, the members of the class of '83 have gone their different ways. Seventy-one percent of the class are males in full-time employment, while sixty-two percent are females in full-time employment.

(2) Another quote in a newspaper

Metropolitan rail fares are likely to increase by 100% (e.g., from \$3.20 to \$6.40) while some country fares will increase by as much as 300% (e.g., from \$12 to \$36)

(3) 'I won't make anything, but I won't lose.'

A bookshop usually marks its books up 25% on the original cost price. After a while, it is clear that some detective novels are not selling, so the bookseller decides to mark down their present selling price, by 25%. 'I won't make anything, but I won't lose', she says.

(4) In 1989 a report argued:

A policy of positive discrimination has allowed the number of women in university professional positions to rise by 60%. The number of men in similar positions over the same period of time has risen by only 6%. It is clearly time to revert to a fairer policy. (Tout & Johnson, 1995, p. 322)

In order to interpret these situations and judge their accuracy a rich conceptual percent schema is required to enable: identification of the whole; changes to the whole upon increase; the multiplicative nature of percent increase; and the point of reference for percent comparisons to be made. Development of percent sense then, is a key aspect of instruction in percent.

The Essence of Percent

In order to take percent instruction beyond practise of written computational procedures is to identify the basic meaning of percent. The difficulty in defining percent is the fact that it takes on different meanings in the various contexts in which it is used (Parker & Leinhardt, 1995). As described by Parker and Leinhardt (1995), percent can be a *number* when it is written in an equivalent fraction or decimal form; percent can be a *comparison* in the part-whole fraction sense (e.g., if a candidate receives 35% of the votes, this percent is the subset of people who voted for this candidate compared to the total number of votes cast); percent can be a *ratio* comparison, where the comparison is between two distinct sets (e.g., there are 400% more boys than girls); percent can be a *statistic* when data is reduced to manageable form for interpretation (e.g., a state's employment rate of 8.5% is compared to the national average of 10%); and percent can be a *function* when amounts are calculated according to a stated percent (e.g., interest rates, discounts). However, despite the fact that percent meaning changes according to the situation in which it is used, the basic essence of percent is proportionality. In summary of the multi-meanings of percent, Parker and Leinhardt (1995) stated that "the

common thread woven through all these descriptions is that percent is an alternative language used to describe proportional relationships—a language that is unique, concise and provides a privileged notation system” (p. 444) and further that it is “fundamentally a language of privileged proportion which simplifies and condenses descriptions of multiplicative comparisons” (p. 472).

Percent Instruction from a Proportional Perspective

Considering a proportional approach to percent, the simplicity of representing the three percent problems as statements of proportion using the proportion equation $\frac{a}{b} = \frac{c}{d}$ can be seen, as in the following examples:

- 17% of the 247 students in the school ride a bike to school; how many is that?
 $\frac{17}{100} = \frac{x}{247}$
- 42 of the 247 students in the school ride a bike to school. What percent is that?
 $\frac{x}{100} = \frac{42}{247}$
- 17%, which is 42 students ride a bike to school. How many students in the whole school? $\frac{17}{100} = \frac{42}{x}$

Expressing these percent situations as statements of proportion enables all percent-related situations to be represented through a similar structure (Post, Behr, & Lesh, 1988). The difficulty of representing percent situations as statements of proportion is in taking the next step to solve the equation. Typically, the cross-multiply technique is suggested, where the solution is obtained by cross-multiplying and solving for the unknown. The cross-multiply technique is not popular as it is considered a meaningless procedure (e.g., Cramer, Post, & Currier, 1992; Hart, 1981) and is not recommended as a starting point for developing proportional reasoning (Post et al., 1988; Streefland, 1985).

Through consideration of this perspective on percent as proportion situations, and other aspects of teaching and learning percent as suggested in the literature, a program of instruction for percent is described here as a basis for rethinking percent instruction in the middle years (particularly for eighth grade and above) and for considering the meaning/skill dilemma that teachers face in schools (as outlined by Trafton & Thiessen, *this volume*).

A program of percent instruction in which a four-step proportional procedure for percent problem solving was an integral element, has been reported as a means for significantly assisting eighth-grade students' percent problem solving performance (Dole, 2000; 1999b). The program was designed to enable students to operate successfully in the domain of percent through utilising a single method for the three basic types of percent problems found in school textbooks, whilst providing a structure for representing percent situations as statements of proportion, and making efficient use of available school instructional time. The method comprises four steps for (1) interpreting, (2) representing, (3) symbolising, and (4) solving percent problem situations, as described below:

Step 1. Interpreting percent situations

Percent situations contain three elements, which can be identified as being the *part*, the *whole*, or the *percent*. In any percent problem, two elements are given, and a solution requires finding the third. Using the previous three examples, identification of the three elements in each situation would be as follows:

- (i) 17% of the 247 students in the school ride a bike to school; how many is that? Part = ?; whole = 247; percent = 17%
- (ii) 42 of the 247 students in the school ride a bike to school. What percent is that? Part = 42; whole = 247; percent = ?%
- (iii) 17%, which is 42 students ride a bike to school. How many students in the whole school? Part = 42; whole = ?; percent = 17%

Step 2. Representing the problem

A vertical dual-scale number line is used to represent the elements of the percent problem. The structure of the number line is such that the percent elements are always located on the left side of the line and the quantity under consideration is located on the right, thus providing a strong visual image of the *whole* being equivalent to 100%, and the *part* being equivalent to the percentage. The three examples above would be represented as in Figure 1. It can be seen from the representation that the placement of the *percent* and *part* on the number line also assists estimation and reasonableness of solution.

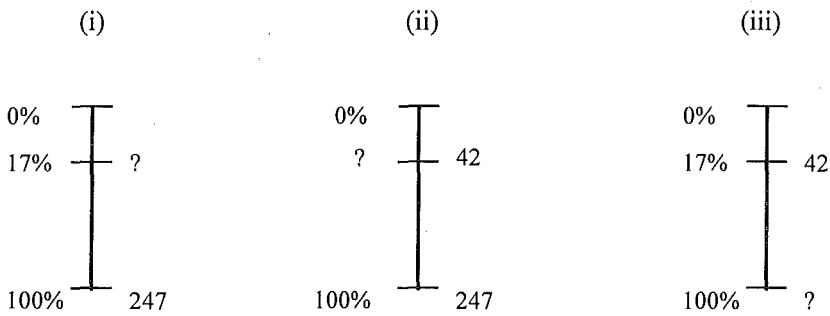


Figure 1: Representing elements of the three percent problem types

Step 3. Symbolising the situation

The percent situation can be written as a proportion equation with the placement of the numbers in the equation directly aligning their placement on the number line:

(i)

$$17/100 = ?/247$$

(ii)

$$?/100 = 42/247$$

(iii)

$$17/100 = 42/?$$

Step 4: Solving the problem

The proportion equation is easily solved using a hand-held calculator and the cross-multiply and divide procedure (cross-multiply the two numbers across from each other and divide by the other one to give the unknown).

$$(i) \quad \frac{17}{100} = \frac{?}{247}$$

$$(17 \times 247 \div 100 = 42)$$

$$(ii) \quad \frac{?}{100} = \frac{42}{247}$$

$$(42 \times 100 \div 247 = 17)$$

$$(iii) \quad \frac{17}{100} = \frac{42}{?}$$

$$(42 \times 100 \div 17 = 247)$$

Conceptual Basis

According to Parker and Leinhardt (1995), one of the difficulties of percent lies in reading, interpreting and defining the relationships within percent problems. In the procedure for percent problems outlined above, identification of the elements in terms of part, whole and percent provides a point of initial access to the problem. In a similar vein to a part-whole schema for interpreting addition and subtraction word problems (Mahlios, 1988; Resnick, 1982; Wolters, 1983), the procedure adopts a part-whole-percent schema. For representing the elements within the percent problem situation, a dual-scale, vertical number line is used. The number line is similar to the comparison scales suggested by Dewar (1984) and Haubner (1992). Representation of elements on the number line can also be seen to reduce error in placement of numbers within the proportion equation. To solve the proportion equation, the 'cross-multiply' procedure is used.

The fourth step of this method is clearly the most contentious step as it can be regarded as a "meaningless application of a rule" (Cramer, Post, & Currier, 1992). Looking at the broader picture, the skill is merely an end-point to bring closure to the problem; the important aspect is that to reach that end-point, students need to interpret and represent the percent situation in a meaningful way. The visual image of the situation enables percent situations to be seen as proportional situations where the quantity is being considered in its relation to a base rate of 100. This method provides students with the means to successfully operate in the domain of percent and thus to build confidence. It also appears to pave the way to promote conceptual understanding of percent in terms of parts and wholes, rather than just practising computational procedures. The following statements, gathered during a research study on implementation of this method with eighth-grade students, highlights this:

I found out that part means percent; well, they mean the same thing ... this way is a lot better than the way we learnt last year, so now I do percent problems a lot easier and quicker.

I find percent easier to work out using the number line. Percent is so easy, it's just part, whole, percent.

One thing I learned today was an easier way of finding the percentage of something. I have used that way instead of my own way because it's so easy.

To students, the method provides access to the domain of percent and is clearly welcomed by them.

The value of the proportional number line method is its effectiveness and efficiency. It is effective in providing a conceptual and computational structure for percent situations and provides the means for students to successfully operate in the domain of percent. Its efficiency is in terms of instructional implementation as it provides students with a pathway to promote percent conceptual knowledge through a proportional perspective without requiring well-developed proportional reasoning skills as a platform. The cross-multiply algorithm is a minor part of the method, but its efficiency is its value, and is applied in context rather than practised in a rote manner. The selection of the cross-multiply procedure in this method is an example of choosing an efficient algorithm rather than discarding all algorithms for the sake of having students invent their own (see discussion by Stacey, *this volume*). The cross-multiply algorithm is valuable in reducing cognitive load (Sweller, 1988) and enabling students to concentrate on more important concepts and principles in problem solving (Noddings, 1990).

A powerful aspect of the number line is that it provides the means to take students beyond the three percent problems to explore and conceptualise increase and decrease situations. The number line enables representation of percent increase and decrease situations, simultaneously revealing the multiplicative and additive structure of percent increase and decrease situations. To show percent increase situations, the number line is extended beyond 100% and thus can be seen to stretch the part/whole notion of percent. Figure 2 shows representation of a percent increase and percent decrease situation on separate number lines respectively for the following two situations: (i) a baby's mass increased 25% in three months from its birth weight of 3156g; and (ii) a shirt costing \$75 was reduced 35% in a sale. For the percent increase situation, the representation shows additively that the baby's new mass is its original mass plus 25%, and multiplicatively that the new mass is 125% of its original mass. For the percent decrease situation, the representation shows subtractively that the shirt is its original price less 35%, and multiplicatively that the shirt is 65% of its original price. #Missing?

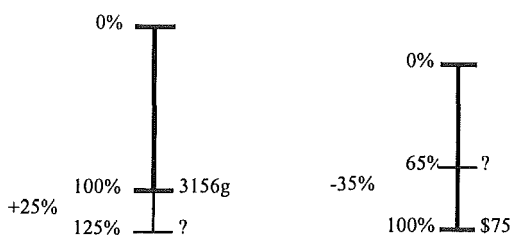


Figure 2: The dual-scale number line representing the additive (and subtractive) and multiplicative nature of percent increase and decrease situations.

Representing situations on the number line can be seen as a means for clarifying naive interpretations of percent increase situations in the real world. For example, consider the following problem:

Jack sells both his motorbikes for \$2000 each and makes a profit of 10% on one and a loss of 10% on the other. Overall, did he break even, make a loss or make a profit?

Intuitively, it would appear that Jack broke even, and this is typically the response given by students and adults alike. With the assistance of the number line, the component elements of the situation can be interpreted and analysed, as depicted in Figure 3.



Figure 3: Representing Jack's motorbike problem

On the first bike, the \$2000 for which Jack sold his bike is 10% less than the amount he paid for it, or 90% of its original price. On the second bike, Jack made a profit of 10% when he sold the bike for \$2000. Thus \$2000 corresponds to 110% on the number line. In both cases, the original price for the bike needs to be found in order to determine whether Jack broke even, made a profit, or made a loss. In this case, Jack made an overall loss of \$40.40 because he originally paid \$2222.22 for the first bike and \$1818.18 for the second bike.

Percent increase situations have traditionally been a difficult aspect of percent (Parker & Leinhardt, 1995). When trialled with eighth graders (Dole, 1999a) the students' journal entries indicated the relative ease with which they could handle such problems, as seen from the statements below:

I was having a bit of trouble with increase and decrease but I now understand them a lot better.

I now know that percent problems are written in two ways. The first is to say how much it went up by and the second is to look at it from the whole.

We never learnt how to do increase/decrease percent problems last year so this is something new to me, but it is not that hard so that is good.

Concluding Comments

In this chapter, the method for solving percent problems was presented as an example of rethinking instruction in percent for the 21st century. The method provides a means for basing the teaching of percent upon a proportional perspective, and also for developing students' understanding of the additive (and subtractive) and multiplicative nature of percent increase and decrease, for promoting estimation and 'percent-sense', and through immersion into the rich domain of percent usage, for broadening students' conceptual understanding of percent as a multifaceted topic. Percent is a confusing and

difficult topic in the middle school curriculum and one in which students have continuously performed poorly. Instruction in percent requires more clarity than that which is currently offered in schools. Analysis of issues associated with teaching and learning percent presented here are for the purpose of extending the debate on computational methods and invented algorithms in school mathematics to the domain-specific knowledge that also must be the result of formal school instruction.

References

- Bennett, A.B., & Nelson, L.T. (1994). A conceptual model for solving percent problems. *Mathematics Teaching in the Middle School*, 1(1), 20-24.
- Boling, B.A. (1985). Sharing teaching ideas: A different method for solving percentage problems. *Mathematics Teacher*, 78, 523-524.
- Brown, G.W., & Kinney, L.B. (1973). Let's teach them about ratio. *Mathematics Teacher*, April, 352-355.
- Cole, B.L., & Weissenfluh, H.S. (1974). An analysis of teaching percentages. *Arithmetic Teacher*, 21, 226-228.
- Cooper, T., & Irons, C. (1987). *Ratio, proportion and percentage in the primary school: Trial materials*. Brisbane: Brisbane College of Advanced Education.
- Cramer, K., Post, T., & Currier, S. (1992). Learning and teaching ratio and proportion: Research implications. In D.T. Owens (Ed.), *Research ideas for the classroom: Middle grade mathematics* (pp. 159-178). New York: Macmillan.
- Dewar, J.M. (1984). Another look at the teaching of percent. *Arithmetic Teacher*, 31, 48-49.
- Dole, S. (2000). Promoting percent as a proportion in eighth-grade mathematics. *School Science and Mathematics*, 100(7), 380-389.
- Dole, S. (1999a). *Percent knowledge: Effective teaching for learning, relearning and unlearning*. Unpublished Doctoral Dissertation, Queensland University of Technology, Brisbane.
- Dole, S. (1999b). Successful percent problem solving for Year 8 students using the proportional number line method. In J. M. Truran and K. M. Truran (Eds.), *Making the Difference: Proceedings of the Twenty-second Annual Conference of the Mathematics Education Research Group of Australasia (MERGA)*. Adelaide: MERGA.
- Dole, S., Cooper, T., Baturo, A., & Conoplia, Z. (1997). Year 8, 9 and 10 students' understanding and access of percent knowledge. In F. Biddulph & K. Carr (Eds.), *People in mathematics education: Proceedings of the Twentieth Annual Conference of the Mathematics Education Research Group of Australasia (MERGA)*. Sydney: MERGA.
- Gay, S.A. & Aichele, D.B. (1997). Middle school students' understanding of number sense related to percent. *School Science and Mathematics*, 97(1), 27-36.
- Glatzer, D.J. (1984). Teaching percentage: Ideas and suggestions. *Arithmetic Teacher*, 31, 24-26.
- Haubner, M.A. (1992). Percents: Developing meaning through models. *Arithmetic Teacher*, 40, 232-234.

- Hart, K. (1981). Ratio and proportion. In K.M. Hart, M.L. Brown, D. Kuchemann, D. Kerslake, G. Ruddock, & M. McCartney (Eds.), *Children's understanding of mathematics: 11-16* (pp. 88-101). London: John Murray.
- Kouba, V.L., Brown, C.A., Carpenter, T.P., Lindquist, M.M., Silver, E.A., & Swafford, J.O. (1988). Results of the fourth NAEP assessment of mathematics: Number, operations and word problems. *Arithmetic Teacher*, 35, 14-19.
- Lembke, L.O., & Reys, B.J. (1994). The development of, and interaction between, intuitive and school-taught ideas about percent. *Journal for Research in Mathematics Education*, 25(3), 237-259.
- Mahlios, J. (1988). Word problems: Do I add or subtract? *Arithmetic Teacher*, 36(3), 48-52.
- McGivney, R.J., & Nitschke, J. (1988). Is-of, a mnemonic for percentage problems. *Mathematics Teacher*, 81, 455-456.
- Noddings, N. (1990). Constructivism in mathematics education. In R.B. Davis, C.A. Maher, & N. Noddings (Eds.), *Constructivist views on teaching and learning of mathematics* (pp. 1-6). Reston, VA: NCTM.
- Parker, M., & Leinhardt, G. (1995). Percent: A privileged proportion. *Review of Educational Research*, 65(4), 421-481.
- Post, T.R., Behr, M.J., & Lesh, R. (1988). Proportionality and the development of prealgebra understandings. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 78-90). Reston, VA: NCTM.
- Resnick, L.B. (1982). Syntax and semantics in learning to subtract. In T.P. Carpenter, J.M. Moser & T.A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp.136-155). Hillsdale, NJ: Erlbaum.
- Reys, R.E., Suydam, M.N., & Lindquist, M.M. (1992). *Helping children learn mathematics* (3rd ed.). Boston: Allyn & Bacon.
- Schmalz, R. (1977). The teaching of percent. *Mathematics Teacher*, 70, 340-343.
- Smart, J.R. (1980). The teaching of percent problems. *School Science and Mathematics*, 80, 187-192.
- Streefland, L. (1985). Search for the roots of ratio: Some thoughts on the long term learning process (towards ... a theory). *Educational Studies in Mathematics*, 16, 75-94.
- Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science*, 12, 257-285.
- Tout, D. & Johnston, B. (1995). *Adult numeracy teaching: Making meaning in mathematics*. Melbourne: National Staff Development Committee for Vocational Education and Training.
- Van de Walle, J. A. (1998). *Elementary and middle school mathematics*. New York: Addison Wesley Longman.
- Weibe, J. (1986). Manipulating percentages. *Mathematics Teacher*, 79(1), 23-26.
- Wolters, M.A.D. (1983). The part-whole schema and arithmetical problems. *Educational Studies in Mathematics*, 14, 127-138.

Part 4

The Teacher's Role

Introduction

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Part 4 points the lens more closely at teachers and professional development – the role of teachers in developing children’s number sense, and the role of professional development in supporting their work.

Sparrow and McIntosh describe a small-scale project that shares the philosophy of the work described by Trafton. Teacher empowerment and reliance on teachers’ professional judgments were central features. The role of the university collaborators was two-fold: to provide external stimulus and support, and to observe, describe and share with other teachers the challenges and realities of the process of classroom change.

Bobis describes a major professional development project in New South Wales whose influence has spread throughout Australia and beyond. She describes the structure of support provided for project schools by Education Department consultants and the flexible nature of the professional development provided. The project, which is assessment-based and centred on the early years of schooling, strikes a fine balance between imposed structure and individual growth by teachers and children.

Askew draws on involvement in a major British research project to illustrate the influence of teachers’ beliefs on children’s numeracy learning. He describes three different orientations towards teaching and learning observed in teachers within the project, and shows that the orientation most associated with student gains in numeracy is that shared and advocated by all the authors of this book.

Developing Number Sense in Classrooms

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What happens in classrooms and to teachers when they decide to develop number sense with children? How does the process of developing number sense proceed? How are teachers and their practice different at the end of the implementation? This chapter will provide insights into these questions by describing aspects of a research project undertaken with two groups of teachers.

Background to the Study

The aims of the project were to:

- provide data from a study of classrooms in which teachers have consciously introduced Number Sense approaches to teaching computation; and
- evaluate a professional development model which empowered teachers to design and implement classroom change related to number sense.

Two Perth metropolitan primary schools were invited to join the project, where one staff member of each school had had contact with the researchers on a previous project. Interested teachers within the schools attended the initial meetings. From these meetings three teachers and one coordinating teacher from School A and four teachers and one coordinating teacher from School B became involved in the project. The classes from each school were almost parallel in age grouping—two Year 3, one Year 4/5, one Year 5, a Year 7, and a Year 5/6/7 combined.

The Developing Number Sense in Education (DENSE) project commenced with a meeting at the end of Term Two (June) and took place during Terms Three and Four (July to December).

A Model for Professional Development

The DENSE project adopted from the start a principle of teacher empowerment as the basis for professional development (Robinson, 1989). In this model emphasis is placed with the teachers to design what happens to them. They establish an agenda for action; they make decisions about what is needed and what will happen next. Thus, the professional development program originates in, and is driven by teachers' concerns, interests and needs. It is situated in the realities of their classroom with their children.

The empowerment paradigm sees a shift from previous models of professional development—which often saw teachers manipulated by the wishes of others—to one of personal control of professional development.

Underpinning the empowerment model adopted was the notion that two essential components for professional growth are needed: teacher experimentation and teacher reflection (Clarke & Peter, 1993). The action research cycle of plan, act, observe and reflect (Kemmis & McTaggart, 1992) was central to the model and was designed to establish the need for reflection and experimentation by the teacher and to set the problem in real and situation specific classrooms.

The stance the researchers adopted also embodied teacher empowerment principles. The familiar researcher and professional development provider role of telling what has to change and how it will happen could not be used. In its place was a role whereby the researchers became convenors of meetings and resource providers. If asked, options for activities or ways forward were offered. They became classroom observers and reluctant partner teachers. Researchers were reluctant because there is an inference that the demonstration or the lesson implies the correct way to work and that the teacher is expected, without evaluation, to copy it—a very familiar professional development experience for many of the teachers. The major role here was one of a ‘fellow worker’ or ‘support teacher’ (Clarke, 1996; Feiman-Nemser, 1992).

Issues related to impediments to change in these classrooms is discussed in more depth elsewhere (Sparrow & McIntosh, 1998). The main obstacles to change were identified as:

- teacher felt expectations,
- time factors due to the need to cover the syllabus content,
- available resources,
- the teachers’ background knowledge and beliefs, and
- the children in the classroom.

Generally over the period of the project, teachers managed to work with and around the constraints and challenges. The next part of this paper will consider changes that took place in teachers and in their teaching while developing number sense with their class.

Number Sense and Teaching

During the project one of the investigators sat in on a number of number sense lessons at his request. His intentions were twofold: first, this would provide the only direct evidence of what was actually happening in the project classrooms; and second, it was hoped that presenting teachers with a detailed account of what the investigator saw and heard, together with some comments, might form a useful form of professional development. Here is a description of one lesson, in the form it was later given to the teacher of the Year 5 class, Kaye.

11.15am. Class on mat at front. Explanation of strategies for $89 + 26$.

Kaye displays pictures of three children [shown calculating $89 + 26$] on the overhead. “Imagine you are the girl, think about the answer to $89 + 26$ and how you could describe what you did to somebody else?” Kaye has a large sheet of paper

attached to the blackboard by the OHP picture. It is marked. "How can we work it out?" After a pause, Kaye asks the children to explain what they did. She writes each explanation in words and symbols on the sheet of paper.

- $8 + 2 = 100$, $9 + 6 = 15$. 115. [When asked why $8 + 2 = 100$ the girl retreated into silence.]
- With fingers. Did 89 add two tens and 6.
- Did it by fives. Added $8 + 2 = 100$, then $9 + 6 = 15$, added $100 + 15$ by 5, 10, 15.
- $80 + 20$, $9 + 6$.
- Imagined the numbers written down vertically, did as a written sum.
- Added $8 + 2$ then added a zero, then $9 + 6$. "How did you know to add a zero?" Silence from boy.
- Changed it to $86 + 92$ and got 115. [Did he mean $86 + 29$? This was not followed up]

11.35am. Discussion of Strategies

"Let's have a look at some of these strategies."

- Discussed why $8 + 2 = 100$, in terms of 8 tens and 2 tens. The child's method was validated.
- Discussed a quicker way of adding 20 to 89 than by counting in ones using fingers. "Count in tens ... 10, 20, 30, ... Count in tens from 3 ... 3, 13, 23, ... Count in tens from 9 ... 9, 19, 29, ... 59 ... Count in tens from 89. [More hesitantly] 89, 99, 109, ..." This felt like a breakthrough for some children.

11.43am. Extension of Activity

Kaye then followed up on preferred methods of calculation: "Before we do some more, would you rather do that calculation [$89 + 26$] with a calculator, in your head, or on paper?" All three methods were advocated by different children. "Can someone give me a calculation they would need a calculator for?"

- " $4 \text{ billion} + 561 \dots 5230 \times 1000$ " were suggested and queried.

11.48am. Activity 2

Ten calculations were revealed on the OHP. "Which of these could you do in your head?" Each was considered with the children being asked to justify their choices.

11.55am. Activity 3

"This example $88 + 88 + 88$. Work it out mentally and then write down the answer and explain how you did it."

12.06 - 12.15pm. Discussion of Activity 3

Several children were asked to share what they had written.

Issues Arising from the Lessons

A mental computation 'lesson' has traditionally been envisaged as lasting about 10 to 15 minutes. In contrast Kaye's mental computation session lasted the entire 60 minutes of the mathematics lesson, although Kaye admitted she had some reservations about this.

I would never do this for so long usually and I would prefer to do it with a small group while others were working.

Again, a traditional mental computation lesson would consist of a series of short computations to which children write answers in silence, emphasis being placed on speed and accuracy. In contrast, Kaye's lesson involved a great deal of class discussion with an emphasis on encouraging children to explain their strategies and justify their answers orally.

I emphasise children's explanations, children listening to the explanations of others, the fact that there are different ways of mentally doing the same calculation, the value of giving your own idea even if you think someone else might think it silly, giving children time to think when I ask them a question.

The purpose of a traditional mental computation lesson, if a purpose was considered by the teacher, might be to increase knowledge of basic facts or to sharpen up the children before the 'main' part of the lesson. Kaye's reason for emphasising mental computation is clearly different.

The area of number sense that I'm dealing with, with my Year 5 class, is Developing Mental Strategies. And the reason I've chosen this is that it seems that children generally have a poor understanding of basic facts and place value. And when they're confronted with a problem they are easily stumped. They don't appear to have a variety of strategies in place to work it through, and even those who appear to be fairly well skilled at pencil and paper computations have trouble when they are confronted with a problem that's posed in different ways.

When reflecting on the value of her mental computation lessons later, Kaye felt that these aims had been realised.

Having a concentration on that and just forgetting about the other parts of number maths for the time being, I'm really amazed how very quickly those children who just had absolutely no idea, I think the first session that we had Alistair was here and there were kids who were just gazing into space, absolutely no idea about how to go about doing that, and other kids who were more able were able to share their strategies, and those children who had no idea have now got strategies to begin to use, and they are choosing other people's strategies, and with some success, which is really terrific.

However Kaye also saw the lessons as having much broader benefits still.

... but the really terrific spin-off is that [their communication skills] have just improved out of sight, as they've realised how precise they have to be in their oral sharing for someone else to understand, because I actually get the person who's listened to someone sharing, they then have to tell that strategy to somebody else and they have to listen very carefully to someone else's strategy and then explain that to another person, so their listening skills have improved, their oral sharing skills and their written skills because they know how precise that has to be for someone else to read it.

Kaye found that the written record of the lesson, while being very informative for the investigators, was both valuable and reassuring for herself also.

I guess copies of the lesson observations are the most useful in terms of perhaps learning teaching strategies used by others. I'm particularly interested in the other Year 5 class at [the other school], especially as we have a common area of focus.

Finally Kaye reflected on the effect on her of the whole process of being involved in the project.

I think this whole project has led me to rethink a whole pile of things to do with my own teaching and I think that's a good thing. I don't think I was doing a bad job before, but I think it's made me [give] children back the ownership for their learning, instead of worrying all the time that I haven't taught this and I haven't taught that. Now I can give them the opportunities to learn things that they need to learn, and that seems to be spreading right across the things that I do. You know, I've been teaching for 23 years, you get a bit stale, so sort of a new approach is very elevating, stimulating for the children ... I'm sort of almost sad that our part in this has come to an end, but I won't end what I'm doing, I shall keep on going and I shall make sure that I continue to find new things to do, maybe I'll move on to something other than mental computation, I'd like to develop fractions.

Number Sense and Teachers

It has been noted in other project reports (Groves & Stacey, 1998; Shuard, Walsh, Goodwin, & Worcester, 1991) that one of the main outcomes of work with teachers was a change in the way mathematics in general and mathematics teaching was viewed. This factor of the project acting as a catalyst for teachers to reappraise their ideas of mathematics and mathematics teaching and learning was also a feature of the present study. In general terms most of the teachers moved from a philosophy and teaching style emphasising standard methods of computation, practice and strong teacher direction to one which allowed the children more freedom to discuss methods and offer alternative and valid ways of working. Two teachers, in particular, illustrate this change. Amanda and Betty (both pseudonyms) will be followed in this part of the paper from the early days of the project to the last meeting in October.

Pressures on Teachers

The image of Betty at the start of the project was one of someone under pressure. Under pressure to cover the content from the syllabus; under pressure to please parents and secondary school teachers; under pressure to provide the right way to do things and to provide a structure for the children so they could reproduce the right way in tests.

...[At] the beginning of the year they [the children] walked in and ... oh yeah fractions, they're all there up on the board, that's what we are doing today, tune out for 50% of them... (Betty, 13th August).

...because I used to walk in and think right, I've got to get all this on the blackboard now, and its got to be there before the siren... (Betty, 13th August).

The emphasis was on the accepted way; a traditional way; and a way that sat well with the perception of the requirements of the school and what teachers do. Issues of institutional tradition and social heritage were particularly strong (Mousley & Clements, 1990).

...I remember drilling the kids on this is how I want the page ruled up, this is how I want this... (Amanda, 13th August).

There was a strong pressure to conform but for many of the teachers they were not aware of other ways to teach number even if they wanted to change their teaching style. The presence of the researchers and other teachers provided the necessary information and support for them to attempt to change.

Fears and Worries

As the teachers moved into the project and started some of the work on number sense in the classroom there were still nagging fears and uncertainties.

I still feel worried about it you know, oh you'll be sick of hearing this I suppose, about that it doesn't feel like I'm doing proper maths so maybe next time I see you Alistair I'll talk this through with you and see if I can feel a bit more relaxed about it (Amanda, reflections tape, 11th August).

The worries were there for many of the teachers throughout the project and will probably remain, even with those teachers who had positive experiences with their teaching.

I still feel a bit worried that they'll move on to the next class and they still won't know what 7 times 8 is.

The principals and the admin always seem to want children to be able to rattle off lots of facts and be really good at algorithms and to do them neatly, and that's how they assess your success in maths.

The feeling was strong that mathematics is about children practising lots of examples of correct methods for calculating and being able to give an instant answer to questions. In many ways it appears from the transcripts that actually being part of the project legitimised being different and trying other things and other ways of teaching mathematics. The presence of the researchers as 'fellow workers', as in the case of Amanda, helped to develop the confidence to try something; to experiment and thus enter the first phase necessary for change in Clarke and Peter's (1993) terms.

By the End of the Project

What was different in the teachers and their mathematics teaching at the end of the project? For many of the members there was a feeling of uncovering children's thinking rather than covering it up or generally ignoring it.

I think this program made us more focussed on how children actually arrive at the conclusions that they do. Some of them have very very long ways of going about things, and through understanding all those little steps that their minds are taking, I think you can help them try other ways. We've found that children listen to other children's methods and then want to try those methods too, so we are not focussing on one particular way of getting at things (Caroline, 13th August).

For Betty, with the Year 7 class who were about to go to secondary school and who had experienced seven years of the right way to do mathematics, things had also changed.

They [the children] say ... what are we doing for maths, and it's really just a general curiosity because it's not on the blackboard any more and I don't know what we are doing for mathematics ... but it's there in the back of my mind (Betty, 13th August).

There was a growing confidence to move into areas of the unknown. Lessons became more adventurous for the teacher and more freeing for the children. The narrow pathway of the right way had been generally abandoned in favour of serendipity.

I know where I'm starting from now, [but] you don't always know how it's going to end... You turn on the ignition ... that's a big change for me; it's a big change for the Year 7s too (Betty, 13th August).

The experience was, at times, a struggle to leave the old and continue with the new.

And when you say to people, well just see what happens, you think aaaghh! I still have to staple my tongue to the roof of my mouth sometimes just to stop myself from telling the kids what ... I find it really hard but I'm getting there and they're starting too (Betty, 13th August).

The move away from a teacher-directed and teacher-led style of teaching to one that was more open and reliant on the children was a huge one for many of the group. Amanda had been teaching for twenty-three years before she made her 'big decision'.

I'm going to stop worrying about doing proper maths because I've decided I'm doing proper maths and it doesn't matter what I thought before, I've just decided I'm going to make this decision and I'm going to go ahead with what I'm doing and to hell with the consequences (Amanda, reflections tape, 22nd August).

Experiences for Children

What changes would children experience in their day-to-day mathematics working with their teachers as a consequence of the project? As mentioned earlier the Year 7 class were no longer always greeted with a blackboard full of 'sums' as they entered the room. Amanda, as a result of her meetings with the staff from the partner school, had decided not to use a maths pad again. Thus the children would not "get bogged down with ruling up pages".

I think there are different ways of doing things and pages and pages of algorithms with ticks and crosses next to them is a bit silly really (Amanda, reflections tape, 17th Sept.).

These children in Year 5 were to move to using scrapbook style with stick-in pages and be required to think about and reflect on their mathematics learning by using a mathematics journal. For Amanda there had been quite a revolution in thinking about mathematics and her mathematics teaching. The project with its dual emphasis of both teacher experimentation and reflection had been effective with Amanda. It was quite noticeable from the data that she was the one who had produced the most taped journals and had done considerable reflecting on her practice with the following results.

I think this whole project has led me to rethink a whole pile of things to do with my own teaching and I think that's a good thing. I don't think I was doing a bad job before... I've given children back the ownership for their learning, instead of worrying all the time that I haven't taught this and I haven't taught that, now I'm focussed on giving them the opportunity to learn things that they need to learn, and that seems to be spreading right across the things that I do, which I think is a good thing, and I feel more relaxed about it (Amanda, reflections tape, 17th September).

Techniques from other Subjects

With the development of a feeling of confidence many of the teachers reported that they were taking the chance to escape the straightjacket of their previous teaching style for mathematics. Now they were starting to experiment with using techniques previously employed in language and other lessons. There was more discussion taking place with a requirement for the children to explain and offer reasons for their choice or for their method.

I was going to say that I'm finding that I'm using more of those skills and techniques that we've used in language, such as brainstorming, classification, those kinds of things I'm using more and more in the mathematical area, so in that sense I'm not treating maths as a separate kind of thing (Joint meeting, 13th August).

Children were not only experiencing a different style of mathematics teaching but also being expected to do more and take ideas further. Often now, the somewhat artificial constraints of the syllabus content for the particular year level was being lifted.

...they amaze me sometimes how far they can actually go, because you haven't put a limitation on them (Joint meeting, 13th August).

Similar findings to this are also reported in the Calculators in Primary Mathematics project and the Calculator Aware Number project (Groves & Stacey, 1998; Shuard et. al., 1991). In these cases the use of a calculator and the requirement to leave the teaching of formal algorithms for computation were the agents of freedom.

Issues of Coverage

Time factors and the need for syllabus coverage added to the teachers' worries. Mathematics was seen to be in nice blocks of time and with experience a teacher could judge the amount of maths to be covered and practised within that space. Now with an emphasis on number sense activities and more open ended situations this was not so easy to achieve. The oral aspect of a lot of the activities presented teachers with worries.

I find that in maths we start on something and by the time we've finished fiddling about with it the time has disappeared. And then I get all worried and I think, heck, they won't be able to do anything. They are having fun but will they be able to do anything with the products of this fun?

I think one of the things that worries me is that I am going to end up with kids who really love maths but still can't do it! You know it's really good doing all this number sense but I still have to do long multiplication ... it's juggling the two things (Amanda, 17th June).

The last part of Amanda's comment suggests that the issue of number sense is seen as separate to the mathematics that must be covered. The emphasis is on coverage and remembering the correct procedure rather than on learning. An awareness that number sense teaching could be integrated into helping the children multiply two numbers has not at this point of the project occurred to Amanda. The problem then arrives that the method adopted by the child to achieve the correct answer may not be the 'right one'. Part of a way forward happened in the thinking of Betty later in the project when she announced that she was going to test what the children knew and could do rather than testing their ability to set sums out correctly. One of the teachers who professed to be quite confident with mathematics, having done mathematics at college, spoke of the need for mathematics to be relevant. Later she noted that after having undertaken an activity related to number sense development, she found it difficult to see how it was useful. For her it appeared that useful meant being able to calculate using the standard methods. The number sense activities were very nice but were not relevant to performing the right methods for calculating.

On the Positive Side

But not all was worry and anxiety. The combination of teacher experimentation and teacher reflection and being empowered to do something different had positive effects for the teachers and in particular for many children. There was a feeling of growing confidence in mathematics teaching and learning.

The changes that I've noticed in myself are that I feel much more relaxed about the actual things I'm doing, and just, the lessons are more fun, and they are less formal so far, and I think the children, they're very happy (Amanda, 11th August).

Amanda's reflections summed up the general feeling about mathematics teaching and how it had changed as a result of the catalyst of trying out number sense activities.

Conclusions

The progress of the project in developing number sense in classrooms and enhancing teachers' knowledge of number sense was slow, erratic and often moved in unexpected directions. Gradually, enthusiastically and openly, teachers began to share—warts and all—happenings, problems and most of all successes as they worked with both the idea and the practicality of teaching mathematics with a number sense philosophy. Teachers can and do begin to use their own judgement rather than wait for 'experts' to tell them what to do and think. The empowerment model with the 'fellow worker' support forced this to happen. It also allowed for the main features of experimentation and reflection in the Clarke and Peter (1993) model to take place.

The mere fact that the 'fellow worker' was convening meetings and leading the project in the schools appeared to give experimentation and risk-taking legitimacy. Where the two schools came together for a meeting with reporting, planning and discussion, there was an openness in reflection-on-practice. This happened easily and naturally within the group, whereas the act of reflecting as an individual for the taped journal was sporadic or absent for many of the participants.

There appeared to be some change not only to classroom practice with number sense ideas, but also to teachers' perception and knowledge of teaching and number sense concepts during the life of the project. One would hope that changes continued to be implemented and developed after the closure of the DENSE project.

References

- Clarke, D. & Peter, A. (1993). Modelling teacher change. In B. Atweh, C. Kanes, M. Carss, & G. Booker (Eds.), *Contexts in mathematics education*. Proceedings of the 16th Annual Conference of the Mathematics Education Group of Australasia, (pp. 167-175). Brisbane: MERGA.
- Clarke, D.M. (1996). The changing role of the mathematics teacher. *Journal for Research in Mathematics Education*, 28, 278-306.
- Feiman-Nemser, S. (1992). *Helping novices learn to teach: Lessons from an experienced support teacher*. (Research Report No. 91-6). National Center for Research on Teacher Education, Michigan State University.

- Groves, S. & Stacey, K. (1998). Calculators in primary mathematics: Exploring number before teaching algorithms. In L. Morrow & J. Kenny (Eds.), *The teaching and learning of algorithms in school mathematics*. Reston, VA: NCTM.
- Kemmis, S. & McTaggart, R. (1992). *The action research planner*. Geelong, Victoria: Deakin University.
- Mousley, J. & Clements, M.A. (1990). The culture of mathematics classrooms. In K. Clements (Ed.), *Whither mathematics* (pp. 397-407). Melbourne: Mathematics Association of Victoria.
- Robinson, I. (1989). The empowerment paradigm for the professional development of teachers of mathematics. In N.F. Ellerton, & M.A. Clements (Eds.), *School mathematics: The challenge of change* (pp. 269-283). Geelong, Victoria: Deakin University.
- Shuard, H., Walsh, A., Goodwin, J. & Worcester, V. (1991). *Calculators, children and mathematics*. London: Simon & Schuster.
- Sparrow, L. & McIntosh, A. (1998). Teachers building number sense amid the challenges of change. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times*. Proceedings of the 21st Annual Conference of the Mathematics Education Group of Australasia (pp. 532-540). Brisbane: MERGA.

Number Sense and the Professional Development of Teachers

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The notion of teaching mathematics for understanding and for meaningful learning to occur is by no means a new one (Brownell, 1935; Heibert, 1984; Skemp, 1989). However, it was not until the 1980s that the term ‘number sense’ was first coined. The continued emphasis and increasing recognition of its importance to the mathematical development of students has resulted in curriculum documents around the world stressing the need for students to develop a good sense of number (Australian Education Council, 1991; Japanese Ministry of Education, 1989; National Council of Teachers of Mathematics, 2000). Such a change in focus warrants, inevitably, changes in classroom practice. Unfortunately, a dichotomy often exists between what is espoused in curriculum documents and what actually happens in practice. The fact that “high levels of efficiency” in computation remain the determining factor of most classrooms means that significant changes to classroom practices have not occurred (McIntosh, Reys, Reys, Bana, & Farrell, 1997, p. 5).

This chapter is based on the outcomes of a large-scale, long-term professional development project and the impact it has had on teachers, their classroom practices and their students in regard to the development of number sense. The Count Me In Too Project (New South Wales Department of Education and Training, 2004) emphasises the development of meaningful mathematical strategies and concepts in young children rather than on the transmission of rote procedures. The project is one example of how educational systems can support teachers in effecting needed changes not only to the way they teach mathematics but also to the content they emphasise in their classrooms. One aim of this paper is to show how changes to classroom practices, no matter how slight they might be, can result in more meaningful mathematics being learned. Such practices need to take account of children’s intuitive mathematical strategies and concepts and challenge them to reach for more sophisticated levels of understanding through challenging tasks that require reflection and problem solving.

Count Me In Too—Classroom-based Professional Development

Count Me In Too (CMIT) is a professional development initiative of the New South Wales Department of Education and Training (NSWDET). While initially focusing on number in the first three years of school (Kindergarten to Year 2), it has gradually incorporated content appropriate to older grades and to the space and measurement strands of the syllabus. Its primary aim is “for teachers to better

understand children's mathematical strategies and their development from less sophisticated to more sophisticated strategies" (Stewart, Wright & Gould, 1998, p.557). The program has its origins in the theory and methods of the Maths Recovery Program (Wright, Stanger, Cowper & Dyson, 1996) incorporating aspects such as the Learning Framework in Number (Wright, 1998) and the Schedule for Early Number Assessment (NSWDET, 2004). The project, which commenced in 1996, initially involved 4 DET mathematics consultants and over 35 K-2 teachers from 13 schools across NSW. Since that time, the project has continued to expand each year so that by 2003 nearly 1700 primary or central schools throughout NSW had been involved in the program. CMIT has also had considerable impact beyond the NSW government school system. It has been adopted by many non-government schools in the state and by government systems in other Australian states and territories. Aspects of the program have been used by school systems in the United States, United Kingdom, Papua New Guinea and New Zealand.

The project offers funds to release teachers from teaching and to provide classroom support as they as they learn new strategies for teaching, to assess and analyse children's mathematical thinking, using the SENA and LFIN. Independent studies of CMIT's impact on teacher development repeatedly indicate that teachers consider the benefits to themselves and their students far outweigh any negative aspects of the program such as the time needed to conduct diagnostic interviews or to produce new resources (e.g., Bobis, 2003). Hence, the program's implementation has gathered momentum, not through the insistence of educational authorities, but by the commitment of hard-working teachers and their supportive school administrations who have come to realise the benefits of the program.

Rather than being a packaged program, CMIT is a continually evolving school-based initiative that involves a close liaison between the regional consultant, a school-based CMIT facilitator and a team of teachers at each school. A major outcome of the project is the establishment of collegial groups, where professional dialogue relating to mathematics flourishes and where teachers not only devise, share, adapt activities and teaching strategies, but support one another in overcoming the difficulties associated with risk-taking ventures in the classroom. These two aspects of the project—work-based learning and collegiality—are well acknowledged as being crucial factors to the success of other professional development projects (Bobis, 1998; Retallick & Groundwater-Smith, 1996).

The work-based model of professional development operating in CMIT schools varies from school to school, but generally involves a mathematics consultant or a school-based facilitator working alongside teachers for a couple of hours each week. Consultants and facilitators help teachers assess the mathematical development of children, and assisting with the planning and implementation of developmentally appropriate and meaningful experiences. While evaluations of the program repeatedly indicate that CMIT significantly increases teachers' knowledge of children's cognition, particularly in regard to the arithmetical strategies they use (e.g., Bobis, 1999a; 2003), the main difference for most teachers working in the project is often the *way* mathematics is taught. There is much more focus on children's solution strategies, on reasoning, reflection, problem solving and conceptual understanding than on the rote memorisation of algorithmic procedures. So what does an instructional approach that focuses on these aspects 'look like'? What do teachers think about the changes they have made to the way they teach mathematics?

The Way Mathematics is Taught

Changing conceptions about how children learn mathematics based on accumulated research findings and years of documenting 'good' practice have influenced the instructional approach advocated by the CMIT program and adapted by the teachers involved in it. For many teachers, their classroom practices have undergone significant reform—for others, it has either helped them refine their approaches or reinforced their conviction about what they were already doing as being beneficial to children learning mathematics. The instructional approach is closely linked to the research-based Learning Framework in Number (LFIN) originally developed by Wright (1998) and extended through the influence of a wide range of research in early number learning (e.g., Mulligan & Mitchelmore, 1997).

The LFIN incorporates key components of number concerned with the arithmetical development of young children. Each component is arranged into a series of approximately six predetermined stages or levels of development (Wright, Martland, & Stafford, 2000). These components include:

- building addition and subtraction through counting by ones;
- building addition and subtraction through grouping;
- building multiplication and division through equal counting and grouping;
- building place value through grouping;
- forward number word sequences;
- backward number word sequences;
- number word sequences by 10s and 100s; and
- numeral identification.

During a task-based interview (Schedule in Early Number Assessment, or SENA) designed to elicit a child's most sophisticated strategy, classroom teachers determine where each child might be situated for each component on the LFIN. From initial and subsequent assessments, teachers make decisions regarding learning experiences necessary for individual children and groups of children to help them develop more sophisticated strategies and levels of understanding. Hence, the LFIN provides teachers with a type of 'map' for each child—they learn from the children *what* they know and *how* they do it and the LFIN then guides them as to where the children need to develop further. How this development is achieved is where CMIT's approach to instruction is different to the way mathematics has traditionally been taught. While there is a focus on children's existing strategies and knowledge to plan for future instruction, the major emphasis is on mathematics making sense. Perhaps the best way of illustrating how the approach operates is to provide a 'snapshot' of a typical classroom in which CMIT principles are well established. The following scenario is compiled from video-taped lesson excerpts, children's work samples and classroom observation notes made for the purpose of investigating the impact of the program (Bobis, 1999a).

A Classroom Scenario

The teacher of a composite Kindergarten/Year 1, Mrs Sanders, and the district mathematics consultant were team teaching. The children were spread around the room, many working in pairs on assigned tasks, and some in small groups working with either Mrs Sanders or the consultant. The children were grouped according to their strengths and needs as revealed on the SENA, rather than their grade level. Mrs Sanders was working with the less able children while the consultant worked with a group slightly more advanced students.

Mrs Sanders and her group were sitting on the floor at the back of the classroom. She had a bundle of straws with her. She selected a particular number of straws and dropped them into a small bucket in front of the children. The children were informed of the number of straws in the bucket and were asked to calculate the total number once some more straws were added. They could not reach the straws to see, touch or count them.

Mrs S: I'm tired of dropping all these straws, so I want you to *pretend* I'm dropping them in the bucket. So 13 straws (no straws are shown) and this many more (shows three straws) is...

(All children immediately raise their hands to provide the answer. Janice sub-vocalises 14, 15, 16 with her hand already raised.)

Edward: 14.

Janice: I got 16.

Mrs S: Let him work it out. Count with me ... 13 (holds up one straw at a time)
14 ... 15 ...

Children: 16.

Mrs S: Good. Let's try another. 17 and this many more (holds up three straws).

(All children raise their hands immediately to respond. Sam moves his fingers and sub-vocally counts-on from 17. Janice and Greta also sub-vocalise as they count-on from 17, but do not use their fingers to keep track of the numbers.)

Greta: 20.

Mrs S: Yes. (They check the answer together by counting-on from 17 and use the straws to keep track of the numbers.)

(The process is repeated for increasingly more difficult numbers: 23 and 3 more, 35 and 4 more.)

In this activity the children were being encouraged to count-on from numbers other than one with the assistance of concrete materials. They were using their knowledge of forward number word sequences to help them count-on from the starting number provided by Mrs Sanders. At the point where the transcript starts, the teacher encouraged the children to make a conceptual leap in their learning. That is, the first bundle of straws was no longer presented and they had to *pretend* it existed, holding the number stated by the teacher in their heads while they counted-on three more. The teacher was scaffolding their learning towards a more abstract concept of number by using an activity in which the children were familiar. However, the most difficult aspect of the task—the counting-on—was still represented concretely.

All of the children were noted to have sub-vocalised while counting-on and two were observed using their fingers to keep track of their counting. The sub-vocalisation

and finger counting are typical of children at this early stage of arithmetical development and help identify children experiencing difficulty with the counting-on strategy. Once the initial bundle of straws was no longer required by the children, Mrs Sanders was free to select larger and more difficult numbers from which the children were asked to count-on.

A little later in that same lesson, in another part of the classroom, two Year 1 boys were seated at their table playing a game. The mathematics consultant was close to their table, but working with another group of children on the floor. The children had an up-side-down ice-cream container and ten Unifix cubes. The two boys took turns to cover their eyes while their partner removed some cubes and hid them under the ice-cream container. On one occasion, Ben took three cubes and hid them from his partner, Scott.

Ben: Open your eyes.

Scott: (Pointing to each block he sub-vocalised as he counted them. After counting 7 cubes, he made an immediate response.) 3!

(He lifted the container to check but did not count them. He was satisfied that he was correct.)

Consultant: (Hearing Scott's reply, the teacher asked him to explain how he determined the answer.) How come there's 3?

Scott: 'Cause there's 7 up there and (touches each cube) 8, 9, 10. Your turn. Close your eyes. (Ben closed his eyes and Scott placed 5 cubes under the container). Open.

Ben: (Counted sub-vocally the 5 remaining cubes as he touched each one. He responded immediately after counting.) 5.

Consultant: How did you know there were 5?

Ben: Because there were 5 on top and 5 and 5 is 10.

The children involved in this activity were practising their number combinations to 10, but with only one part of the whole visibly represented by concrete materials. Both boys counted by ones to determine the number of cubes visible, but neither needed to count-on to calculate the remainder hidden under the container—they seemed to know what number was needed to make 10. Ben's knowledge of number facts was illustrated when he immediately recognised '5 and 5 is 10'. Scott, however, used counting-on to justify his answer to the consultant. These children had moved beyond the counting-on strategy to calculate number combinations to 10, but still used it to explain how they derived their answer or to check if unsure of an answer. The concrete materials were still necessary as evidenced by the fact that both boys needed to count one-by-one when the number of cubes was greater than 4.

Towards the end of group work time, two Year 1 children were sitting at their table rolling three dice. The three numbers on the dice were added and if the total corresponded to a numbered card on the table they collected the card. They had almost completed the activity with only a few numbered cards remaining on the table. Mrs Sanders had set her group to work independently and was now moving around the class to watch how other groups were working. She stopped to observe Jason and Leah complete their activity.

Jason: (Rolls the dice.) 5 and 4 is 9 and one more is 10. (No sub-vocalisation or counting of fingers is required, but he raised his eyes to look at the ceiling for an instant while adding 5 and 4 to make 9.)

Oh! No, 10. Your turn.

Leah: (Rolls the dice and sub-vocalised 5, 6, 7 as she pointed to the dots on the die.) 5 and 2 is 7 and another 2 is 9. No, 9. I can't go.

Jason: (Rolls the dice.) 1 and 1 is 2 and 2 is 4. Yeah there's a 4. Game finished. (Calling to teacher.) Mrs Sanders, we're finished.

(The two children continue to roll the dice despite there being no numbered cards left on the table.)

(Jason rolled the dice.) 6 and 5 is 11 and another 5 is ... (looked at ceiling for an instant) ... 16.

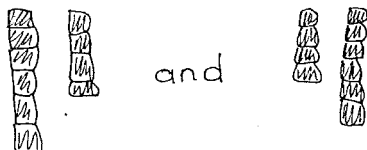
Mrs S: How did you know 11 and 5 is 16?

Jason: Because 5 and 1 is 6 and then add on 10 and that's 16.

Jason and Leah were using slightly different strategies to help them add three numbers. Jason was using quite a sophisticated mental computation strategy that relied on his understanding of parts and whole number combinations. He did not need to count-on, nor did he use his fingers to keep track of his mental calculations. He remembered that '6 and 5 is 11' but did not know what 11 add 5 makes. He applied a strategy that is not unlike the procedure used in the vertically arranged addition algorithm—the two numbers in the ones column were added first and then 10 more was added. Leah, on the other hand, still needed to count-on to determine '5 and 2 is 7'. However, she did know that '7 and another 2 is 9' indicating that she was already committing some number facts to memory.

To conclude the lesson, the whole class came together to discuss and share their experiences. The children playing the ice-cream container game were asked to talk to another pair of children and then to write about any 'discoveries'. Figure 1 presents Janice's (Year 1) reporting of a discovery she made while playing the game. She discovered the commutative property of addition. That is, regardless of the order in which numbers are added, the answer will always be the same, so 6 plus 4 is the same as 4 plus 6. Understanding this property of addition (and multiplication) assists students in learning their basic number facts—almost halving the number of facts to be learned.

Today I learnt that if you
have $6 + 4$ you can turn it
around. Like this $4 + 6$. It is
the same. See



Janice
Well that's what I learnt!

Figure 1: Janice's recording of her discovery while playing with Unifix cubes

In this scenario we see Mrs Sanders, with the help of the district mathematics consultant, working with a wide range of students in her class—a lower ability group who needed a lot of scaffolding to help them correctly apply the counting-on strategy, a more advanced group of children spontaneously applying the same strategy and building-up their familiarity with number combinations to ten, and another, even more advanced group, sharing the discovery of a more sophisticated strategy (the commutative principle) also for number combinations to ten.

While this particular scenario does not illustrate *all* the different teaching strategies that typify the CMIT approach to teaching and learning mathematics, we can see a number of practices. Most significantly, the students are provided with activities that are appropriate to their level of knowledge and strategy development. Hence, children often work with other children in small groups or pairs according to their needs rather than their grade or age. In this way, knowledge and skills are continuously built-upon as no child is exposed to activities too far beyond or below their capabilities. Children are encouraged explicitly to utilise increasingly more sophisticated strategies within their zone of development—they are not ‘taught’ them as meaningless procedures or when it is obvious they are not developmentally ready to understand them. Strategies, such as counting-on, are linked to the naturally occurring strategies of the children and to their developing sense of number.

Another characteristic of the approach emerging from the scenario, is the lack of emphasis on the teaching of formal paper and pencil written algorithms. This does not mean that they are not dealt with in CMIT classrooms, however, there is a much greater emphasis on mental computation, problem solving and the use of concrete materials for longer periods of time. It is also demonstrated in the scenario, that children are constantly encouraged to justify and reflect upon their answers by asking them to verbalise their thinking strategies and to record their discoveries in writing or in their drawings. The ability of children to apply, explain their thinking strategies and to develop autonomy in solving non-routine problems is greatly enhanced by teachers consistently asking them to justify their answers in this way.

What do the Children Learn?

While it is difficult to report succinctly or systematically on the outcomes of the children in regard to the development of number sense in such an approach, especially since the development of understanding and meaning is such a dynamic process, the benefits to the children are perceived by teachers, consultants and parents almost immediately the CMIT program begins operating. On a free response item in a questionnaire sent to all teachers participating in the 1996 program, 30 percent of teachers responding commented “on the children’s increased confidence” in dealing with unseen problems, on their “enjoyment and enthusiasm to solve problems and use large numbers” more than their students of the past (Bobis, 1999a, p. 16). During interviews conducted in the same study, one teacher commented that she was now giving children “the opportunity to come and tell me things that they’ve discovered rather than me telling them...” and that as a result of this new approach, along with other new teaching strategies she had been experimenting with, the children were taking more control over their own learning (p. 28). Other teachers commented on the fact that the new assessment techniques had allowed them to know better the mathematical ability of all the children so that they could “take them to the edge as often as” they could while

still keeping them feeling secure (p. 23). Many teachers felt that providing appropriate challenges to children of various abilities meant that each child not only understood more mathematics, but they all now enjoyed learning mathematics.

While the perceived benefits of CMIT to children can be justified by classroom observations, samples of children's work, and data collected from questionnaires and interviews of teachers involved in the program, they can also be justified quantitatively. Recent and ongoing investigations that compare the mathematical achievement levels of children in the CMIT program to those not involved support the qualitative evidence of previous studies indicating that children's mathematical achievement is significantly improved (e.g., White & Mitchelmore, 2002). For example, one evaluation of the program involved two groups of children—a Kindergarten and a Year 1 class from two neighbouring schools in a pre-test/post-test situation. Prior to the implementation of CMIT in the experimental school, test results indicated that there were no significant differences between the two Year 1 groups, but that the Kindergarten control group (children from the school not undertaking CMIT) performed significantly better than the Kindergarten experimental group on the Forward and Backward Number Word Sequence components. However, both the Kindergarten and Year 1 experimental groups performed significantly better than their control group counterparts at the post-test phase (Bobis & Gould, 1999). The significant advances made by the Kindergarten and Year 1 experimental groups are clear evidence of the positive impact Count Me In Too can have on the mathematical achievement of children involved in the program.

In short, besides the potential academic improvements, children learn that mathematics can be meaningful and more exciting. They also seem to develop a greater confidence in their own abilities to solve even unseen problems. The children learn to take more control for their own learning as they become confident to use 'invented' strategies that make sense and are consistently required to explain and justify their thinking to themselves and others through classroom discussions and their own recordings.

What do the Teachers Learn?

It has been shown over the past few years of the CMIT operation that, among other things, teachers learn to raise their expectations of what young children can do and how they can do it. They learn that children are capable of developing their own strategies for solving problems and they learn how to give children the opportunity to 'discover' things rather than 'telling' them everything. At first, many teachers are surprised by the level and richness of the strategies children are capable of developing once they are provided with opportunities to develop more autonomy and adaptability in the learning situation. As one teacher commented:

I now focus on computational methods in that I offer more than one way... Now I ask them instead of telling them. I give them time to think... (There is) much more sequential development of teaching number with a greater commitment to using a variety of strategies to encourage thinking mathematically. (Bobis, 2003, p. 14)

Responses to open-ended items on a questionnaire and comments from teachers during interviews revealed that, generally, teachers felt that not only had their content knowledge of mathematics been extended, but their depth of understanding of how children learn mathematics had increased as well. For example, one teacher commented that "it's scary what I didn't know" about how children learn mathematics (Bobis, 2003).

Others commented on their new knowledge about the importance of “arrays to teach multiplication and division” and their “better understanding of place value”. Some also mentioned how their involvement in the program had impacted on the way they themselves perform mental computation and how they now “pass on this to the children” (p. 13).

Hence, besides increasing their own knowledge of mathematics, teachers involved in CMIT learn a great deal more about how children learn mathematics and about how much they can do. They learn how to ask questions that will not only assess children’s learning, but also encourage further learning to occur. They learn how to establish a classroom environment in which reflection and mathematical thinking are valued. More significantly, they learn how all the different components of their professional knowledge—their knowledge of content, of how children learn mathematics and of the pedagogical strategies useful for enhancing mathematical understanding—are all interconnected. As one teacher commented: “I have learnt so much about how children learn... My knowledge links to *what* the children learn and *how* they learn ... it is all interlinked” (Bobis, 1999b, p. 27). The ultimate outcome of such insights is the establishment of classrooms in which mathematics makes sense—both to the children and to their teachers.

Conclusion

The Count Me In Too Project is one example of how educational systems can support teachers in effecting needed changes not only to the way they teach mathematics but also to the content they emphasise in their classrooms. Some of the key features of the approach espoused by the project were presented via a scenario and accompanying discussion. We saw how the approach emphasises classroom practices that foster the development of meaningful mathematics in young children—such practices focus on children’s solution strategies, on reasoning, reflection, problem solving and conceptual understanding rather than on the rote memorisation of algorithmic procedures.

While the positive outcomes of CMIT have been emphasised here, it is not an intention to give the impression that such changes in teachers’ classroom practices occur ‘overnight’ or without some trauma. Realities of the change process, including the identification of potential barriers, have been a focus of program evaluations in the past. For example, Bobis (2003) confirmed previous reports that most teachers starting CMIT are concerned, and may experience considerable stress, as a result of the time needed to conduct individual assessment interviews with their children or to make the resources needed by the program. For these and other reasons, some have elected not to join the program or to continue with it after its introduction. However, teachers can resolve such problems with the support of their administration and colleagues. One group of teachers commented that by working together they were able “to manipulate our day so that we could get some time to do the assessments” and that subsequent assessments were easier because they had become more efficient with conducting the interview and were more “organised with their time and classes” (Bobis, 1999a, p. 22).

As part of CMIT’s dynamic nature, the program is continuously evolving and expanding to higher grades and to new content areas in the curriculum. However, an obstacle to its sustained impact, has been a mismatch between the program’s focus on the development of flexible mental computational strategies and a curriculum still emphasising the early introduction of standard algorithmic procedures. In 2002 a new

Mathematics K-6 Syllabus (Board of Studies, NSW, 2002) was introduced that focuses on the development of increasingly more efficient thinking strategies and delays the introduction of standard pencil and paper algorithms by nearly two years. The fact that the notion of teaching for number sense is now consistent with CMIT objectives and with a reformed curriculum makes its existence more sustainable.

References

- Australian Education Council (1991). *A national statement on mathematics for Australian schools*. Melbourne: Curriculum Corporation.
- Board of Studies, New South Wales (2002). *K-6 mathematics syllabus*. Sydney: Author.
- Bobis, J. (1998). Partnerships with practising teachers: Action research in practice. In N. Ellerton (Ed.), *Current issues in mathematics education* (pp. 18-38). Perth: MASTEC, Edith Cowan University.
- Bobis, J. (1999a). *Report of the valuation of the Count Me In Project 1996*. Sydney: New South Wales Department of Education and Training.
www.curriculumsupport.nsw.edu.au/maths/countmein
- Bobis, J. (1999b). *Count Me In Too: A case study of implementation*. Sydney: New South Wales Department of Education and Training.
www.curriculumsupport.nsw.edu.au/maths/countmein
- Bobis, J. (2000). *The impact of Count Me In Too on the professional knowledge of teachers*. Sydney: New South Wales Department of Education and Training.
www.curriculumsupport.nsw.edu.au/maths/countmein
- Bobis, J. (2003). *Count Me In Too: An evaluation Stage 2*. Sydney: NSW Department of Education and Training. www.curriculumsupport.nsw.edu.au/maths/countmein
- Bobis, J., & Gould, P. (1999). *The impact of Count Me In Too on the mathematical achievement of Year 1 students*. In J. M. Truran and K. M. Truran (Eds.), *Making the difference*. Proceedings of the 22nd Annual Conference of the Mathematics Education Research Group of Australasia. Adelaide: MERGA.
- Brownell, W. (1935). Psychological consideration in the learning and the teaching of arithmetic. In *The tenth yearbook of the National Council of Teachers of Mathematics* (pp. 1-31). New York: Teachers College, Columbia University.
- Heibert, J. (1984). Children's mathematics learning: The struggle to link form and understanding. *The Elementary School Journal*, 84, 496-513.
- Japanese Ministry of Education (1989). *Curriculum of mathematics for the elementary school*. Tokyo: Printing Bureau.
- McIntosh, A., Reys, B., Reys, R., Bana, J., & Farrell, B. (1997). *Number sense in school mathematics: Student performance in four countries*. Perth: MASTEC, Edith Cowan University.
- Mulligan, J., & Mitchelmore, M. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education*, 28, 309-330.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

- New South Wales Department of Education and Training (2004). *Count me in too: A professional development package*. Sydney: Author.
- Retallick, J. & Groundwater-Smith, S. (1996). *The advancement of teacher workplace learning*. Wagga Wagga, NSW: Centre for Professional Development in Education, Charles Sturt University.
- Skemp, R. (1989). *Mathematics in the primary school*. London: Routledge.
- Stewart, R., Wright, B., & Gould, P. (1998). Kindergarten students' progress in the Count Me In Too project. In C. Kanes, M. Goos & E. Warren (Eds.), *Teaching mathematics in new times*. Proceedings of the 21st Annual Conference of the Mathematics Education Research Group of Australasia, (pp. 556-563). Brisbane: Mathematics Education Research Group of Australasia.
- White, P., & Mitchelmore, M. (2002). *The impact of Count Me In Too on Year 3 basic skills test numeracy scores*. Sydney: New South Wales Department of Education and Training.
- Wright, B. (1994). A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics*, 26, 25-44.
- Wright, B. (1998). An overview of a research-based framework for assessing and teaching early number. In C. Kanes, M. Goos & E. Warren (Eds.), *Teaching mathematics in new times*. Proceedings of the 21st Annual Conference of the Mathematics Education Research Group of Australasia, (pp. 701-108). Brisbane: Mathematics Education Research Group of Australasia.
- Wright, B., Martland, J., & Stafford, A. (2000). *Early numeracy: Assessment for teaching and intervention*. London: Paul Chapman Publications/Sage.
- Wright, B., Stanger, G., Cowper, M., & Dyson, R. (1996). First-grader's progress in an experimental mathematics recovery program. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning* (pp. 55-72). Adelaide: AAMT/MERGA.

Learning Together: Effective Teaching of Numeracy

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Following are two examples of contrasting mathematics lessons dealing with equivalent fractions.

Example 1

A class of nine-year-old students was working on equivalent fractions. The teacher drew a diagram on the board to demonstrate a means of converting $\frac{1}{2}$ into quarters. She explained that quarters are the fraction to convert to, so the students would need to draw a rectangle divided into four equal parts, as in Figure 1a. She then explained that since half is required, two of these parts needed to be shaded in, as in Figure 1b. "So, a half is equivalent to two quarters," explained the teacher. "On the other hand," she continued, "we could just look at the numbers on the bottom of the fraction. I have to multiply 2 (pointing to the 2 on the bottom of the $\frac{1}{2}$) by 2 to make 4 (pointing to 4 on the board of a yet denominator-free quarter fraction), so I multiply the 1 (pointing to the 1 on the top of the $\frac{1}{2}$) by 2 also. So we get $\frac{2}{4}$, which is the same as we got when we drew the diagram" (see Figure 1c).

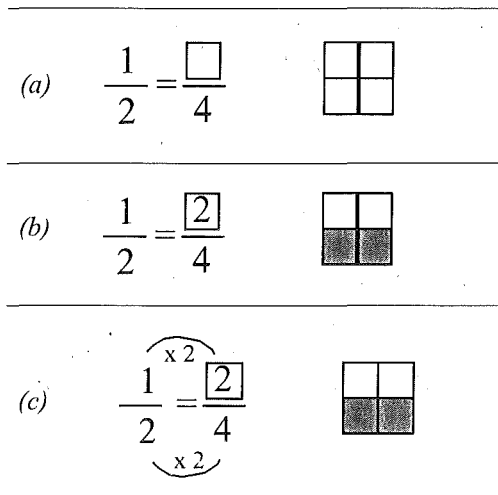


Figure 1: Teacher demonstration of equivalent fractions

The students were given a number of fractions to convert into equivalents and told they could either use the diagram method or the multiplication method. They set off to work on their own. Once individuals had done a few examples using the diagram, the teacher moved around the class suggesting that it would be quicker to use the other method. The teacher helped students who were making errors by re-explaining both methods. At the end of the lesson, the teacher went over the answers on the board, reminding the students of the 'rule' to multiply the top and the bottom by the same number.

Example 2

A class of ten-year-olds. The teacher had put a chart on the white board that had columns for fractions, decimal fractions, percentages and ratios. One value had been entered in each row and the students were working in pairs to figure out how to convert values from one form of representation to another. Some of the values used equivalences that they were already familiar with, for example 0.5 or 25%. Others they did not know already; for example, $\frac{3}{8}$, 0.65, and they were having to work out their own methods of conversion. The pairs used a variety of methods but discussed these with each other and they knew that they should check their answers using a different method. While they were working on the task the teacher moved around listening to their explanations, taking notes on the different methods pairs were using and occasionally joining in the discussion. As they began to complete the task the teacher brought the class together. Pairs were invited to come to the board to provide the answers and explain the method of calculation used. The teacher selected the pairs on the basis of the notes taken and included some students whose reasoning was inaccurate. The other students were attentive to these explanations. More efficient methods were offered and errors dealt with in a supportive manner either by the teacher or other students. Finally the class discussed the sort of contexts where the different representations would be used.

While dealing with similar aspects of the mathematics curriculum, these two examples illustrate different ways in which teachers might set up lessons and the experiences from which students might learn. In this chapter I argue that such differences may be based on different beliefs that the teachers have on the nature of the relationship between teaching and learning, and examine research evidence suggesting that different beliefs may not only affect styles of teaching but also students' learning outcomes.

Effective Teachers of Numeracy

Exploring teachers' beliefs, however informal or unarticulated, about the relationship between teaching and learning was one aspect of the 'Effective Teachers of Numeracy' project carried out at King's College by myself and colleagues Margaret Brown, Valerie Rhodes, Dylan Wiliam, and David Johnson and funded by the Teacher Training Agency¹.

¹ The views expressed here are those of the author and should not be interpreted as representing the views of the Teacher Training Agency

The beliefs of the teachers in the project appeared to be significant not only in terms of what they did in the classroom, but also in terms of children's learning outcomes. This chapter explores some of these issues. Anyone wishing to read more about the project should refer to our report (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997).

The principal aim of the Effective Teachers of Numeracy project was:

- to identify key factors which enable teachers to put effective teaching of numeracy into practice in the primary phase.

In developing this aim, we had two initial issues to clarify. First, what exactly was meant by 'numeracy' and, second, how were we to identify 'effective teaching'? At the time of the project the term numeracy was in little use in the English education system and we could find no agreed definition of numeracy. We therefore decided to adopt a definition that was broad enough to encompass the ability to calculate accurately but also go beyond that to include a 'feel for number', and the ability to apply arithmetic:

Numeracy is the ability to process, communicate and interpret numerical information in a variety of contexts.

With regard to effective teaching, many people in mathematics education—researchers, inspectors, teachers—would claim to know what 'good' practice in primary mathematics should look like. However, evidence linking teaching practices with learning outcomes is relatively limited. Research in mathematics education in the United Kingdom largely separates findings on children's learning from those on teaching.

We decided therefore to base our identification of effective teaching on some measure of children's actual learning gains, rather than presumptions of 'good practice'. Once we had identified classes where students appeared to be learning more mathematics than in other, comparable, classes, we could go about exploring what practices appeared to be most effective in promoting this learning.

We chose to measure children's learning by looking at the gains for individual classes over part of a school year. Specially designed tests of numeracy were given to 90 classes, spanning ages from 5- to 11-year-olds. The first assessment was carried out towards the beginning of the autumn term 1995, and then repeated at the end of the spring term 1996 (The 5-year-olds were only assessed on this second occasion).

On the basis of the students' test results, average gains were calculated for each class, thus providing an indicator of 'teacher effectiveness' for the 90 teachers in our project. We then set about looking for factors associated with these class gains. One set of data that we examined was questionnaire responses that each teacher provided and which included details on qualifications, experience and styles of teaching. To examine what it might be that made some teachers more effective (in terms of mean class gains) than others we looked at the relationship between mean class gains and:

- lesson organisation: whether the teacher taught the class as a whole class, in groups or set individual work;
- initial teaching qualifications: how the teachers had trained and the subjects they had specialised in;
- mathematics qualifications: the level to which they had studied mathematics as a subject in its own right;

- resources used: the commercial materials that teachers used to support their teaching;
- experience: how long they had been teaching and the range of classes and ages taught; and
- professional development: how much training the teachers had been involved with since they qualified.

Some of our findings were surprising in that they challenge popularly held beliefs about what makes a teacher effective. For example, we could find no association between mean class gains and:

- class organisation—whole class, groups, individual;
- initial teaching qualifications; or
- experience.

Some of our teachers whose classes made the most gains did a lot of whole class teaching but so did some of the teachers with low mean class gains. Similarly, group or individual work was used by teachers across the spread of gains. The same published mathematics schemes were used by highly effective and comparatively much less effective teachers. The types of qualifications that teachers had were not a good predictor of the mean gains that their classes made; nor was the length of experience as a teacher.

Perhaps our most surprising finding was that there was a slight negative association between mean class gains and the teachers' mathematics qualifications: the better qualified in mathematics that the teachers were, the lower the mean class gains. However, this finding should not be interpreted as indicating that one does not need to know much mathematics in order to teach it. A subset of 18 of the 90 teachers was studied in greater depth, including carrying out a detailed interview about mathematics and their knowledge of it. Analysis of the responses to this interview did indicate that the teachers whose classes made great gains did themselves have a rich understanding of the mathematics that they taught. Taking the questionnaire and interview findings together suggests that simply being well qualified in mathematics may not indicate whether or not a teacher has the sort of mathematical knowledge required to teach it effectively.

In contrast, the amount of continuing professional development in mathematics education that teachers had undertaken was a better predictor of their effectiveness: there was a positive association between mean class gains and the amount of in-service training that the teachers had engaged in.

If aspects such as styles of classroom organisation and levels of mathematical qualification did not determine effectiveness, then what did? To try and understand in greater depth the factors affecting learning we turned instead to more detailed case study data that we collected on 18 teachers in the project. This data included notes taken from at least three observations of these 18 teachers followed by three extended interviews with each of them. The interviews covered several aspects of teaching and included exploring why the teachers had set up the lessons the way they had, their understandings of mathematics, their views on why students were more or less successful in mathematics, and what actions the teachers took as a result of this.

Through analysis of the interviews and observation data, the main aspect that seemed to make a difference was the teachers' beliefs about the nature of the relationship between teaching and learning.

Impact of Teacher Beliefs on Student Attainment

On the basis of the average gains made by each class the teachers were put into three groups of 'highly effective', 'effective' and 'moderately effective'. In order to try and find out what contributed to the different gains made by different classes we looked at how our case study teachers were distributed across these categories.

The initials of the pseudonyms chosen for the teachers as listed in Figure 2 are the same for teachers from the same school. Thus, for example, Anne, Alan and Alice all taught in School A. Two case study teachers of five-year-olds (Claire and Frances) are not included in the table since they could not be readily identified according to effectiveness as their classes were tested only once, so gains could not be calculated.

Highly effective	Effective	Moderately effective
Anne	Danielle	Beth
Alan	Dorothy	Brian
Alice	Eva	Cath
Barbara	Fay	David
Carole		Elizabeth
Faith		Erica

Figure 2: The case study teachers and levels of effectiveness

In examining the beliefs and understandings of these teachers we looked at the ways that they acted and talked about the triadic relationships between teachers, students and mathematics, as depicted in Figure 3. From this triad, three orientations—clusters of beliefs about teaching and learning—emerged: 'Discovery', 'Transmission', and 'Connectionist'.

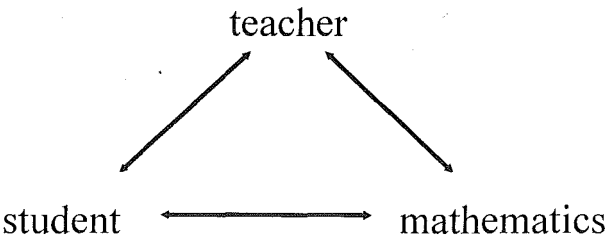


Figure 3: The teaching triad

'Discovery' Orientation

Teachers who displayed evidence of this orientation towards teaching and learning placed more emphasis on the student-mathematics link than the other two links.

Student-Teacher

A discovery orientation emphasises the responsibility of the learner in coming to know mathematics. This leads to valuing student independence above direct teaching. Emphasis on the student as an independent learner carries the presupposition that what is to be learned has not been taught. Having to explicitly teach something is seen as less successful than the students learning it for themselves. Alternatively, if a student fails to learn something, then this may be seen as the result of a lack of readiness rather than inappropriate teaching.

Student-Mathematics

Within the discovery orientation, prior understandings were seen as important determinants of what students are 'ready' to learn. Teachers indicating this orientation spoke as though there is a natural order in which students develop concepts and so progress, and rates of learning are determined by this more than by teaching practices. Teachers displaying a discovery orientation focused on affective aspects of learning mathematics: to learn effectively students needed to be motivated to learn and they had to be able to work independently. Activities were therefore structured around these aims and justified on the basis of being enjoyable and set up to be 'fun'. A heavy emphasis was placed on practical work with the conventions of the subject subordinated to understanding. For example, in one lesson observed, a group of low attaining seven-year-old students were working on doubling. This was done entirely within the context of using bricks to find the answers, even though in conversation with the students it became clear that they could use their knowledge of simple doubles—for example, double four, to double numbers like 400 or 4000.

Teacher-Mathematics

For the discovery-oriented teacher, her role is primarily to set up activities and learning experiences that will facilitate the students' independently finding out about the mathematics. So her main emphasis in thinking about teaching the mathematics is in terms of finding ways to engage student interest, rather than consider the nature of the mathematics to be learned. This quote from one of the teachers in the study largely sums up this orientation:

Well we try and make sure that any work we are giving them is appropriate to their ability, sort of get them to the concepts at the appropriate times, trying to encourage their independence in choosing apparatus they may use when they are doing different bits of maths work, trying to arrange the classrooms so the children can work without an adult, the child can go and take the tools necessary to do the work and already make practical use of the classroom so the children can develop in that way.

'Transmission' Orientation

Teachers who displayed evidence of this orientation towards teaching and learning placed more emphasis on the teacher-mathematics link than the other two links.

Student-Teacher

The transmission orientation is marked by a clear separation of teaching and learning, with the emphasis on the teaching. Teachers disposed towards this orientation see students as dependent upon the teacher for gaining access to mathematics. The transmission-oriented teacher regards herself as primarily responsible for the learning. The student's role in the class is subordinate to that of the teacher. The first example given at the beginning of the chapter would be a typical 'transmission' style lesson with the emphasis on manipulating symbols and the re-teaching of the same techniques if the students did not appear to understand. This quote from one of the teachers about what makes some students more or less successful sums up this position:

A lot of them learn by rote ... one who needs extra help, I will stand behind him when he is doing it and actually work with him for a long long time ... so they (the weakest) get a lot more of my help. Minus minus something, really I teach it by rote... These two are very weak... They have to learn it by rote ... if the child gets a low mark it's probably my fault not the child's.

Student-Mathematics

Within a transmission orientation mathematics is seen as rule and procedure driven. Mathematics is a metaphorical set of objects to be passed on, a body of knowledge to be committed to memory. The transmission orientation focuses on learning in terms of students' ability to retain mathematical ideas—the evidence for this coming from whether the examples worked through are correct. A heavy emphasis is placed on the symbolic and notational aspects of mathematics—students setting out work 'correctly' (i.e., in line with the accepted conventions) is a major part of the practice of doing mathematics and a main source of assessment. Great emphasis is placed on students following mathematical procedures; and the correctness not only of answers, but methods of solution as well.

Teacher-Mathematics

Planning for teaching within a transmission orientation means attending in the main to what is to be taught, not what has been learned. This means the teacher has to have a good knowledge of how to break the curriculum down into step-wise pieces that can be taught sequentially. The prior understandings of students are of little interest. The lesson described earlier, in which the teacher was explaining how to calculate equivalent fractions, illustrates this. One girl interpreted the diagram in Figure 1 on the board incorrectly (it shows how to convert halves into quarters by drawing a block of four squares and shading in two). The interpretation that the girl made was that in order, say, to convert two thirds into sixths she had to draw a block of six squares and shade in three for a third and then the second three for the two thirds! When the teacher noticed this girl's working she stood shaking her head and saying, quietly, "I cannot see what you have done. What have you done?" At no point did she ask the girl to explain her method. Finally the teacher said, "I think you've done enough shading in. Do the rest the other way."

'Connectionist' Orientation

The connectionist orientation encompasses a view of teaching and learning that attempts to reconcile 'teach' and 'learn' rather than treat them as opposites. The connectionist oriented teacher focuses on all three of the bonds in the teaching triad: student-teacher, student-mathematics and teacher-mathematics.

Student-Teacher

The connectionist-orientated teacher aims to develop students' learning while acknowledging the role that teaching plays in this. This does not simply mean teachers paying attention to learners. The connectionist orientation means that the teacher has a strong sense of herself also as a learner, constantly learning about students. Interest in what students have previously learned, how to make sense of students' interpretations of the lessons and how this might be taken into account in planning and teaching is typical within the connectionist orientation. One of the teachers summed up the importance of establishing good relationships with the students:

I can't work with the child unless I am able to have some toehold as to what the child's strengths and weaknesses are. I can test a child, I can, in a formal setting, but I find it so important to be able to communicate with the child on a one-to-one level and to have the child be open and honest with me.

Student-Mathematics

The connectionist orientation draws on mathematics as a network of connections. The connections between different aspects are common features of lessons taught within this orientation. The second example at the beginning of the chapter is typical of this in the way that fractions, decimals, ratios and proportions were not treated as separate topics but related to each other. Other examples would include lessons where addition and subtraction or multiplication and division were taught together. Thus activities represent the complexity of mathematics rather than fragmenting the curriculum into discrete topics. Further to this, multiple representations of mathematical ideas are used. For example, in a lesson on place value we observed the students had to move between expressing numbers in symbols, with base ten blocks and through placing counters in one or other of two hoops designated as 'tens' and 'ones'. The importance of social interactions in establishing shared meanings is implicit in the connectionist orientation. This is shown through interactions with students where understandings are shared, providing time for students to explain their understandings while still providing alternative methods and explanations. As one teacher expressed it:

Children are so mysterious... You just need to talk with them and for them to explain the mechanics, the thinking of what they are doing and don't make any assumptions ... I just say don't worry I'll show you a different way tomorrow. I will go out and rack my brains ... or I might ask a child. I ask the children as well to explain to each other.

Teacher-Mathematics

Planning for teaching within a connectionist orientation means attending both to what has been learned as well as to what is to be taught. This means the teacher has to have a good knowledge not only of how students learn mathematics in general and the understandings of the particular students being taught, but also knowledge of effective activities and ways to explain aspects.

Orientations and Student Learning

Labelling orientations as connectionist, transmission or discovery are as ideal types, since no single teacher is likely to hold a set of beliefs or practices that precisely matches those set out within each orientation. Teachers, like anyone, will develop personal orientations that draw on a variety of beliefs or practices.

However, analysis of the data revealed that some teachers were more predisposed to talk and behave in ways that fitted with one orientation over the others. In particular, Anne, Alan, Barbara, Carole, Claire, Faith, all displayed characteristics indicating a high level of orientation towards the connectionist view. On the other hand, Brian and David both displayed strong discovery orientations, while Beth, Cath and Elizabeth were characterised as transmission orientated teachers.

Other case study teachers displayed less distinct allegiance to one or other of the three orientations. They held sets of beliefs that drew in part from one or more of the orientations. For example, one teacher had strong connectionist beliefs about the nature of being a numerate student but in practice displayed a transmission orientation towards beliefs about how best to teach students to become numerate.

Looking at the grouping of the teaching into these three orientations alongside the previous classification of the teachers into having relatively high, medium or low mean class gain scores suggests that there may be a relationship between student learning outcomes and teacher orientations, as illustrated in Figure 4.

Orientation	Highly Effective	Effective	Moderately Effective
Strongly connectionist	Anne Alan Barbara Carole Faith		
Strongly transmission			Cath Elizabeth
Strongly discovery			Beth David
No strong orientation	Alice	Danielle Dorothy Eva Fay	Brian Erica

Figure 4: The relation between orientation and effectiveness

Discussion

At the time of writing, teachers in England are being encouraged to adopt a particular style of lesson, one that has three distinct parts: an oral and mental starter, a main teaching section and a plenary. Throughout the lesson there is to be a lot of ‘interactive whole class teaching’. While there is no doubt that this has increased access to mathematics for many primary school students, it is also clear that there are still wide variations in the learning brought about. I suggest that we need to look beyond surface features of lessons to understand why different teachers have different impacts. Examining orientations towards teaching mathematics can help us understand why practices that have surface similarities may result in different learner outcomes.

For example, while all the teachers in the study employed some whole-class question and answer sessions, the nature of the interactions with students within such sessions varied according to the teachers' orientations. The transmission-oriented teachers tended only to focus on right answers—assumed to be based on methods that they had previously taught. Any incorrect answers were not explored. The discovery-oriented teachers were interested in methods as well as answers and valued the students producing a range of methods, but the discussion did not extend to considering whether or not some methods were more or less efficient than others. The connectionist-oriented teachers also valued the range of methods of solution that the students came up with, but they would also discuss the relative merits of different methods and, when appropriate, suggest further methods that the students had not considered.

Expecting teachers to adopt new practices may result either in the practices being adapted to fit with existing beliefs or in limited take-up of the practices themselves. As other research on developing teaching has demonstrated, exhorting teachers to adopt particular practices without helping them develop a deep understanding of the principles behind these practices does not in itself lead to raised standards (Alexander, 1992). In a current project at King's College (the Leverhulme Numeracy Research Programme) we are exploring this issue in more detail. An example from this project illustrates the difficulty of changing beliefs and practices.

As part of England's national numeracy strategy, teachers are encouraged to elicit methods of solution from pupils and to work on effectiveness and efficiency. In one observed lesson the students were working on simple shopping problems, presented orally by the teacher and with each child writing his or her solution on an individual white board to hold up and show the answer. While the students were getting the answers correct the teacher would invite two or three to explain how they arrived at that answer. However the discussion did not include exploring whether or not any particular methods were more efficient than any others. In fact the students seemed to be treating the discussion as a challenge to see who could come up with the most unusual method. As the questions got more difficult and several of the students began to show incorrect answers, the teacher ceased to ask how they worked them out and instead went on to show how to use paper and pencil methods to work out the answers. This actually resulted in more of the students making mistakes as they tried to use the teacher's method rather than one that made sense to them personally.

Teachers may find it helpful to examine their belief systems and think about where they stand in relation to these three orientations. In a sense the connectionist approach is not a complete contrast to the other two but embodies the best of both of them in its acknowledgement of the role of both the teacher and the students in lessons. Teachers may therefore need to address different issues according to their beliefs: the transmission orientated teacher may want to consider the attention given to student understandings, while the discovery orientated teacher may need to examine beliefs about the role of the teacher.

Anna Sfard (1998) suggests that there are two main ways in which we talk about, and consequently think about, learning: learning as a process of acquisition and learning as a process of participation. There is much talk about teaching that assumes an acquisition model: *delivering* the curriculum, *raising* standards, whether or not students have *got it*.

While not wishing to dispute the fact that the overall aim of teaching should be that students have acquired some mathematical knowledge, the sorts of lessons that they have participated in on the way to acquiring that knowledge will have a dramatic impact on the sort of knowledge they end up with. Perhaps the thing that most distinguished our connectionist teachers was the ways in which they tried to make their classrooms 'communities of learners' (Rogoff, 1995), where everyone learned from everyone else. One of the teachers expressed it thus:

I think that as each year goes on you learn more and you discover that you need to know more as each year goes on rather than when you start. So I think that your learning curve is disproportionate in that you will think it would get easier but in fact as each year goes on you see different ways of doing things that will make it better in different ways of extending things. I think that you are learning each year as you go on and there is more and more to learn.

References

- Alexander, R. (1992). *Policy and practice in primary education*. London: Routledge.
- Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997). *Effective teachers of numeracy: Report of a study carried out for the Teacher Training Agency*. London: King's College, University of London.
- Rogoff, B. (1995). Evaluating development in the process of participation: theory, methods, and practice building on each other. In E. Amsel & A. Renninger (Eds.), *Change and development: Issues of theory, application and method*. Hillsdale, NJ: Lawrence Erlbaum.
- Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, 27(2), 4-13.

Conclusion

Beyond Written Computation: Some Answers and even more Questions

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Introduction

The authors in this book have not collaborated in their writing. Their experience is spread over (at least) three continents, diverse backgrounds and differing experiences of educational settings. They write from a variety of standpoints: professional development, university research, classroom experience, and collaborative initiatives. However, their collective voice is remarkably consistent and compelling, but their tone is not haranguing or a call to unthinking conformity. They are not pushing a new bandwagon—quite the reverse. Their request to us, as to children, is based on a respect for individual decisions based on information and reflection. They essentially say:

It is more important that you think, than you think like us.

Since the Rottneest Island conference—the initial stimulus for this book—the world and mathematics education has moved on. New debates have been instigated as new documents and research findings are presented. In the United States, The National Council of Teachers of Mathematics has published *The Principles and Standards for School Mathematics* (NCTM, 2000), and in England the *National Numeracy Strategy* with the embedded Numeracy Hour (DfEE, 1999) has developed further. On a more global scale, the *Third International Mathematics and Science Study* (TIMSS) (Hollingsworth, Lokan, & McCrea, 2003) has been published. Its tables of achievement in mathematics by country have been analysed. Positions on league tables have been defended; others have been used to further arguments for a particular style of teaching; and much has been said about adopting teaching styles exemplified by countries that appeared near to the top of the table.

This chapter is an attempt to gather up some of the common threads of the various chapters, and to focus on some of the issues as they appear to us. The issues, questions, dilemmas and partially formed ideas are gathered under several headings. Appearing again under these headings are the central themes of the book—the development of conceptual understanding; the importance of the teacher role; and analysis and action based on critical incidents. Invariably there is overlap between sections as happens with many attempts at a simple categorisation.

Approaches to Teaching

Over twenty years ago Plunkett (1979) noted that

... a large amount of time is at present wasted on attempts to teach and to learn the standard algorithms and that the most common results are frustration, unhappiness and a deteriorating attitude to mathematics. (p. 4)

There are still a lot, and we would suggest far too many adults and children who have a hatred and fear of mathematics based on their experiences of trying to learn it. Beginning teachers and teacher education students vow they will not teach in the same way and produce the same results. Inevitably they do. The 'cycle of tradition' in mathematics teaching is difficult to break, with much of its myth and practice unquestioned. Many schools are conservative places with ways of working that soon engulf even the most confident and enthusiastic new teacher. Cries from many parents of 'they teach maths differently now' are not true. Mathematics teaching in most classrooms today is based on the bedrock of speed recall of number facts and remembering and reproduction of standard procedures for written computation. Teaching mathematics in many cases has not changed for over fifty years. In fact, McIntosh (1979) noted that advice and recommendations for improving teaching of mathematics given over the past 100 to 150 years had not been heeded and brought into common classroom practice. Mathematics teaching in many primary and elementary classrooms remains as transmission of digit manipulation through a textbook medium or from teachers who experienced the same, did not and still do not understand what is happening, and hated mathematics for the most part of their school experience.

More recently, in the report of the Third International Mathematics and Science Study, a number of Australian mathematics educators and teachers were asked to comment and reflect on the findings contained in the Australian report. After viewing the report and video evidence McIntosh noted:

My second reaction, a depressing one, is that, if these videos and data represent fairly normal current practice in these countries ... then there are a lot of pretty boring, artificial, low-level, irrelevant, mentally stifling lessons being delivered around the globe in the name of Year 8 mathematics, and it is not surprising that so many adults don't want to know anything more about mathematics after they leave school. I have a feeling that if people in 100 years' time view these videos, they will wonder how such rubbish was allowed to continue for so long. (p. 106)

The debate that has been taking place in a number of places over the past 80 years or so relating to children developing conceptual rather than procedural understanding of number continues. Skemp (1976) added to the discussion with his suggestion of developing relational understanding and produced material to support this philosophy. Few have followed him into print with a similar viewpoint.

Connecting appears to be a key idea in children's learning especially in mathematics. The report from King's College, *Effective teachers of numeracy* (Askew, Brown, Rhodes, Wilam, & Johnson, 1997) noted the superiority of what they called 'connectivist' teachers—those teachers who helped children to connect new knowledge to knowledge they already possessed. The research team found that neither exposition nor discovery styles of teaching were as effective in helping children learn mathematics.

A conclusion one might draw from the findings of this study is that teaching approaches based primarily on textbook use, where children complete workbook pages or photocopiable sheets, appear unable or unwilling to accommodate such a connectivist teaching style. Movement beyond written computation will be limited as long as the main teaching style in primary mathematics classrooms is based largely around the mass-produced textbook that panders to and reinforces a limited, conservative and outdated view of what constitutes effective mathematics teaching, particularly in the area of computation.

There appears to be some movement away from an over reliance on textbooks or commercial mathematics schemes with a further impetus for sense making in mathematics by children and associated change in teaching style by teachers. In the UK teaching styles have been questioned by among others Her Majesty's Inspectors of Schools. They noted in a recent report (DfEE, 2002), and often have done so in recent years, that teachers rely too much on children working through textbooks and photocopied sheets on their own as their main teaching strategy. By using these teaching methods teachers were limiting opportunities for children to develop and use their own methods of thinking and recording. The implementation of the National Numeracy Strategy, while some may consider it to be restrictive, has helped many teachers move away from a purely textbook approach to teaching mathematics.

There is a tendency for many children to feel that they must use standard written methods even when they are able to reach the correct answer mentally or by their own written strategy. When children employ their own methods of recording calculations these are often used as aids to personal understanding and thinking, and not merely as a means of recording for someone else to demonstrate a grasp of a particular procedure. Effective teaching helps children derive their written methods from their knowledge of mental strategies of calculation and their ability to explain how they reached their answer. That is, teachers need to help children link their mental methods with methods that are written and within this to move progressively from informal, and possibly lengthy methods, to those that are more formal and compact. What is missing at present are research data to show how this might be done for all operations. Quite rightly teachers ask the question, *How do you help children link mental methods with efficient and effective personal written recording?*

Associated with an emphasis on teaching methods that require children to think are strategies designed to have children make decisions and choices. Decisions not only about what and how to record but also about what sort of computational method (mental, written, or electronic) is needed for a calculation. These are the decisions of the everyday world outside school. If a mental or written method was selected then children would also have to decide which of their strategies in mental or written computation was appropriate for the particular calculation and context. Recent work by Swan (2002) has highlighted reasons for computational decisions made by children and the influences that teachers, among others, have on this process.

In this volume, Trafton and Thiessen (2004) discuss the use of problem-based learning as an example of a teaching strategy whereby children have to make decisions. In this case mathematics learning is seen as an activity rather than a system of ready-made rules and procedures to be encountered and remembered. Here young children are expected to operate in a mathematical way in a mathematical environment within the classroom as well as outside of it. As Trafton and Thiessen demonstrate, ordinary things become a problem if children are not presented with a standard procedure to calculate them. But we have to be careful that we do not replace thoughtless application of

procedures with blind searching for ways that are already known in some rose-tinted belief that children will 'discover' mathematics for themselves. Effective teaching is so much more than that. Gravemeijer and Van Galen (2003) urge teachers to help and guide children to see the mathematics present in situations and to connect it to what they know about operating in a mathematical way.

Mental calculation, whereby children calculate examples such as $29 + 67$ in their heads, has gained momentum and positive results in many classrooms around the world as reported by, among others, Angilheri (2000), McIntosh, Bana, & Farrell (1995), Reys, Reys and Hope (1993), and Thompson (1997). This appears to be a way of bringing sense making into the mathematics classroom and into the teaching of number in particular. It seems that mental computation is a key to understanding. More information and recommendations are still needed to help influence teachers and administrators, especially in the areas of mental strategies for multiplication and division. Teacher resources are also needed to counter balance classrooms and publications that remain overwhelmed by a speed recall view of mental mathematics.

In many ways, teachers need to view children in different ways to those that hold sway at present. For example, if children are seen as mathematical thinkers rather than copiers or repeaters of material within a narrow context, teachers can begin to set more ambitious goals for them to achieve. An approach focussed on skills and procedures only limits children's growth in mathematics and in computation in particular.

The observation, analysis and interpretation of critical incidents also appears to have a key role in comprehending what is happening and for planning a way forward for the learners. The way forward for the learners cannot be planned by a textbook. This is a difficult and sophisticated task for an informed teacher, but only teachers can build on what children know and what they feel is needed next. A one-size-fits-all approach is limited in effectiveness. Teachers are constantly making decisions about pathways for children in developing computational skills and understandings. This is a difficult role to implement but it has to be adopted if understanding is seen as a goal of computation. It cannot be abrogated to a book.

Many beliefs about teaching have been long held and reinforced by time and are therefore difficult to question or change. There appears to be a tenuous link between manipulative materials and symbolic forms of mathematics, especially in the area of number and computation. Children often fail to make the connections between the practical and the abstract. For many years limits were placed on the content to be taught based on ages and grade levels. Children in the early years of school only met numbers to twenty even though it is now known that many of them were capable of comprehending and using much larger numbers and more complex ideas. As teachers begin to assess children's entry ability and adopt an outcomes approach some of these artificial limits are being removed.

Approaches to Change and Effecting Change

The 'cycle of tradition' is strong and very resistant to being broken. Children spend up to twelve years in classrooms and experience numerous mathematics lessons. By the time some of them reach the beginning of their teacher education course, they feel they already know how to teach mathematics. Many grew to hate mathematics,

often as a result of how it was taught to them, but once they are in the classroom, they will replicate almost everything that happened to them. Thus, another generation of mathematics fear and hatred is kindled in children.

A cry of 'tell us what to do', is often heard from busy primary teachers as they are met with another report or vague initiative. What to do to develop sense making and understanding of computation, however, is not a simplistic, recipe-like, step-by-step procedure. Kilpatrick, Swafford and Findell (2001) noted that:

effective programs of teacher preparation and professional development help teachers understand the mathematics they teach, how their students learn that mathematics, and how to facilitate learning. In these programs, teachers are not given prescriptions for practice or ready-made solutions to teaching problems. Instead, they adapt what they are learning to deal with problems that arise in their own teaching. (p. 10)

The role of the thinking teacher is vital to the learning of children in all learning areas, but especially so in mathematics. On the other hand, in the United Kingdom a centrally mandated curriculum for mathematics teaching has been set with the implementation of the National Numeracy Strategy (DfEE, 1999). The initiative appears to have raised standards, as measured on Government tests, though not as much as was wanted. At present, there is no extensive evidence to show that such a move will work in the long term. A view that has emerged as part of the debate associated with the development of the National Numeracy Strategy is that a search for improvement by uniform means may lead inevitably to mediocrity and a stifling of creativity as results and teaching styles regress to the mean.

For many teachers new to the profession and the primary or elementary classroom the first year in teaching is stressful and alarmingly complex. Too often they are unsupported and 'left to get on with it' in a sink or swim atmosphere. As a result many potentially good teachers sink and leave the profession while others struggle to keep afloat and never manage to achieve mathematics teaching beyond a splashing, gasping dogpaddle level. Systems for supporting beginning teachers need to be put into place that allow them to think, to break the cycle of tradition, and to move beyond a heavy reliance on unthinking written computation. This is not achieved in busy schools by a large meeting held once a year for all beginning teachers, nor by devolving the responsibility without adequate resources, nor by quickly appointed and untrained mentor teachers.

Sparrow and McIntosh (2004) in this volume have drawn attention to the limited effect of a professional development model for experienced teachers that relies on attending a 'one off session' on an aspect of teaching mathematics. They noted the need—as do Kilpatrick, Swafford and Findell (2001)—to address the mathematical issues confronting the teacher in his or her classroom and to provide support for the teacher to resolve these issues over time. Further research is needed to design and evaluate models to support and develop the mathematics teaching of both new and experienced teachers to allow them and their students to move beyond written computation.

Conclusion

There is much conjecture and opinion on the subject of computation and written computation in particular. Often this opinion is polarised without the support of

appropriate research evidence. More detailed research is needed to inform debate, curriculum documents and classroom practice. The following are a list of questions that appear to need more evidence:

- Is there a sequence in the move from personal mental to personal written to standard or efficient written methods of calculating?
- Should one explicitly teach or develop mental strategies for calculating?
- What strategies do children employ in mental computation for multiplication and division calculations?
- How is it possible to develop confidence, competence and enthusiasm for mathematics in primary school teachers?
- What is the role for calculators in the classroom?
- What are the implications for curriculum if calculators are easily available to children in school and adults out of school?
- What is the potential of calculators in developing number sense and numeracy?
- What might curriculum sequences and progression look like in a calculator-available classroom?
- What is the role of contextual problems in supporting reasoning and facilitating less traditional computational methods?
- What is number sense and what does it mean to primary school teachers and their classroom practice?
- How do you teach for number sense development?
- What are appropriate computational goals for the future?

So what needs to be done to move ‘beyond written computation’ and deliver appropriate computational goals for the future? The answer appears to be simple and complex—both at the same time. It is simple in the sense that all we have to do is to help children understand and make sense of mathematics in general and their work with numbers in particular. It is complex in that we have to change the thinking and negative attitude of many people directly involved with teaching mathematics. We have to exchange the predominant teaching style of textbook and worksheet use, which limits the power of both teachers and children, for something that is built to meet the needs of the local and immediate. We have to define what is ‘basic’ in number learning for the present, rather than hold onto something that is of the past. Nearly three decades ago Girling (1977), suggested that the ability to use a four-function calculator sensibly could be seen as a basic mathematical ability. We have to deliver appropriate computational goals for the future once we have decided what they are. We have to move from ‘one-off’ encounters with new ideas to situations whereby these ideas are revisited and have a real chance to establish themselves in the mind of the learner. To the complexity is added the fact that we do not have adequate data from research to convince educators of its worth and to show teachers how it can be applied in the day-to-day world of the classroom; often with around thirty primary-aged children.

What then lies beyond written computation? In a fantasy world of the near future there would be many or even most children with positive attitudes to mathematics who are adept at manipulating simple calculations mentally, who perform those that are too

difficult for the head sensibly and in an informed way on calculating machines of various sorts or with personal jottings on paper. Written computation will remain but it will be in a different form with a less important role in the primary classroom—one that is more in line with its role in society.

References

- Anghileri, J. (2000). *Teaching number sense*. London: Continuum.
- Askew, M., Brown, M., Rhodes, V., Wiliam, D. & Johnson, D. (1997). *Effective teachers of numeracy*. London: King's College, University of London.
- DfEE (1999) *The National Numeracy Strategy*. Sudbury, Suffolk: DfEE Publications.
- DfEE (2002) *Teaching calculation in primary schools: A report by HMI*. Sudbury, Suffolk: DfEE Publications.
- Girling, M. (1977). Towards a definition of basic numeracy. *Mathematics Teaching*, 81, 4-5, 13-14.
- Gravemeijer, K. & van Galen, F. (2003). Facts and algorithms as products of students' own mathematical activity. In J. Kilpatrick, W.G. Martin & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 114-122). Reston, VA: NCTM.
- Hollingsworth, H., Lokan, J. & McCrae, B. (Eds.) (2003). *Teaching mathematics in Australia: Results from the TIMSS 1999 video study*. Melbourne: ACER.
- Kilpatrick, J., Swafford, J. & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics*. Washington: National Academy Press.
- McIntosh, A. (1981). When will they ever learn? In A. Floyd (Ed.) *Developing mathematical thinking* (pp. 6-11). London: Addison Wesley for the Open University.
- McIntosh, A. (2003). A typical Australian Year 8 mathematics lesson? In H. Hollingsworth, J. Lokan, & B. McCrae (Eds.) (2003). *Teaching mathematics in Australia: Results from the TIMSS 1999 video study* (pp. 106-108). Melbourne: ACER.
- McIntosh, A., Bana, J., & Farrell, B. (1995). *Mental computation in school mathematics: Preference, attitude and performance of students in Years 3, 5, 7 and 9*. Perth: MASTEC, Edith Cowan University.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Plunkett, S. (1979). Decomposition and all that rot. *Mathematics Teaching* 8(3), 2-7.
- Reys, B., Reys, R. & Hope, J. (1993). Mental computation: A snapshot of second, fifth and seventh grade student performance. *School Science and Mathematics*, 93, 306-315.
- Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 26.

- Sparrow, L. & McIntosh, A. (2004). Developing number sense in classrooms. In A. McIntosh & L. Sparrow (Eds.) *Beyond written computation*. Perth: MASTEC, Edith Cowan University.
- Swan, P. (2002). *The computational choices made by students in years five to seven*. Unpublished doctoral thesis. Edith Cowan University, Perth.
- Thompson, I. (1997). Mental and written algorithms: Can the gap be bridged? In I. Thompson (Ed.) *Teaching and learning early number* (pp. 97-111). Buckingham: Open University Press.
- Trafton, P. & Thiessen, D. (2004). Linking mental and written computation via extended work with invented strategies. In A. McIntosh & L. Sparrow (Eds.) *Beyond written computation*. Perth: MASTEC, Edith Cowan University.